TD1

1 Review Questions

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a. Inequality of income across countries was already very wide at the end of the medieval period.

- *Answer*

  False: It started widening after the Industrial Revolution. Before the revolution, growth was almost null around the world. Production was very close to the subsistence level and there were no investments. Technological progress was the catalyst of growth and, since it was heterogeneous across countries, induced an increase in inequality.

2 The Golden Rule for Savings

This problem aims to get you used to the basic logic of the Solow-Swan model. This exercise is a particular case in which we have a Cobb-Douglas production function, population growth and no technology. In the notation of the class, population growth is $n > 0$, technological growth is $g = 0$ and the production function is $f(k) = k^{\alpha}$ with $0 < \alpha < 1$.

Question: Can you get what is the original production function $F(K_t, L_t)$?

Question: What is the interpretation of α? Why is it between 0 and 1?

a. Express the steady-state level of consumption c^* as a function of k^* and the exogenous parameters n, δ and α (but not s).

Question: Before starting, what is the difference between exogenous and endogenous?

First, we have to recall how to express consumption in this model. What is consumed is what is produced minus what is invested, therefore $c = y - sy$. The saving rate s is exogenous, but production is a function of capital $y = f(k)$. We are also reminded that at the steady state $sf(k^*) = (\delta + n)k^*$. This is key to expressing the steady state level of consumption, c^* as a function of the exogenous variables and k^*, but not s:

$$\begin{aligned} c^* &= y^* - sy^* \\ &= f(k^*) - sf(k^*) \\ &= f(k^*) - (\delta + n)k^* \quad \text{By substituting the steady state condition} \\ &= (k^*)^{\alpha} - (\delta + n)k^* \end{aligned}$$

Here we have c^* as a function of k^*, n, δ and α.

b. Use the result to the previous question to find the optimum level of k^* from the point of view of consumption.

This question asks to maximise consumption with respect to capital. We want to answer the question: what is the level of k^* that gives me the maximum c^*? This is a standard optimisation problem. The first order condition requires setting the first derivative of the objective function, in our case consumption in the steady state, to be equal to 0:

$$\frac{\partial c^*}{\partial k^*} = 0 \Rightarrow \alpha(k^*)^{\alpha-1} - (\delta+n) = 0 \Rightarrow k^* = \left(\frac{\delta+n}{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{1}$$

Remember that you can express this quantity differently by playing with the exponent!

Question: What about the second-order condition?

c. Express the steady-state level of consumption c^* as a function of the exogenous parameters only (n, δ, α, also including s).

We have to perform the same operation as before, but without including k^* in the expression for consumption. Therefore, we must first find k^* as a function of the exogenous variables only, to be able to substitute it in the expression for consumption. We start from the fact that $sf(k^*) = (\delta + n)k^*$:

$$\begin{aligned} sf(k^*) &= (\delta+n)k^* \\ \frac{1}{k^*}(k^*)^\alpha &= \frac{\delta+n}{s} \quad \text{By substituting } f(k^*) = (k^*)^\alpha \\ (k^*)^{-1}(k^*)^\alpha &= \frac{\delta+n}{s} \quad \text{Since } \frac{1}{(k^*)} = (k^*)^{-1} \\ (k^*)^{\alpha-1} &= \frac{\delta+n}{s} \\ (k^*)^{\frac{\alpha-1}{\alpha-1}} &= \left(\frac{\delta+n}{s}\right)^{\frac{1}{\alpha-1}} \quad \text{Multiply both exponents by } \frac{1}{1-\alpha} \end{aligned}$$

$$k^* = \left(\frac{\delta+n}{s}\right)^{\frac{1}{\alpha-1}} \tag{2}$$

We can now substitute k^* in the expression for c^*, as it is expressed only as a function of exogenous variables:

$$\begin{aligned} c^* &= (1-s)f(k^*) \\ &= (1-s)(k^*)^\alpha \\ &= (1-s)\left(\frac{\delta+n}{s}\right)^{\frac{\alpha}{\alpha-1}} \end{aligned}$$

Here we have c^* which only depends on s, n, δ and α.

d. Use the result to the previous question to find the optimum level of the savings rate s from the point of view of consumption.

Again, same story as the previous point, but instead of maximising for k^* we do it for s. First, we rewrite c^* to be able to take the derivative easily. Remember that $\left(\frac{\delta+n}{s}\right)^{-a} = \left(\frac{s}{\delta+n}\right)^{a}$. By changing the fraction we had in the previous point we obtain $\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$. We also isolate the variable with respect to which we must take the derivative (only to make the computations easier, there is no need to do this if you are comfortable with derivatives):

$$c^* = (1-s)\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} = (1-s)(s)^{\frac{\alpha}{1-\alpha}}\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

We are now ready to solve the maximisation problem (here I provided very detailed calculations, if you are comfortable with calculus you do not need to write everything as I do here). First, I recall the rules for deriving a product. In general, we have the following:

$$\frac{\partial f(x)g(x)}{\partial x} = f'(x)g(x) + f(x)g'(x)$$

In our case the two functions are $f(s) = (1-s)$ and $g(s) = s^{\frac{\alpha}{1-\alpha}}$. Everything is multiplied by the constant $\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$ which does not affect the derivative (you should remember this fact, if not, ask!). First, we compute the derivative for each of the functions that compose the product:

$$g'(s) = \frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}-1} = \frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}$$

$$f'(s) = -1$$

Then, by applying the general rule:

$$\frac{\partial[f(s)g(s)]\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}}{\partial s} = \frac{\partial[f(s)g(s)]}{\partial s}\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

$$= \left[-1s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}\right]\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

By setting the derivative equal to zero, we can get rid of the constant (equivalent to dividing each side of the equality by the constant itself).

$$\left[-1s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}\right]\cancel{\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}} = 0$$

We are left with the following:

$$\frac{\partial c^*}{\partial s} = 0 \Rightarrow \left[-1 s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha} s^{\frac{\alpha}{1-\alpha}} \frac{1}{s}\right] = 0$$
$$\Leftrightarrow \left[-1 \cancel{s^{\frac{\alpha}{1-\alpha}}} + (1-s)\frac{\alpha}{1-\alpha} \cancel{s^{\frac{\alpha}{1-\alpha}}} \frac{1}{s}\right] = 0$$
$$\Leftrightarrow \left[-1 + (1-s)\frac{\alpha}{1-\alpha}\frac{1}{s}\right] = 0$$
$$\Leftrightarrow (1-s)\frac{\alpha}{1-\alpha}\frac{1}{s} = 1$$
$$\Leftrightarrow (1-s)\alpha = (1-\alpha)s$$
$$\Leftrightarrow \alpha = s$$

The s that maximises consumption is exactly α, the exponent of the production function. This is more or less intuitive, the more your function is relatively productive, as captured by α, the more you save.

Question: Again, what about the second-order conditions for this problem?

e. Comment the results obtained to questions 2 and 4: how are they related?

Consider the expressions (1) and (2) that we computed in points **b.** and **c.**

$$k_1^* = \left(\frac{\delta + n}{\alpha}\right)^{\frac{1}{\alpha - 1}} \tag{1}$$

$$k_2^* = \left(\frac{\delta + n}{s}\right)^{\frac{1}{\alpha - 1}} \tag{2}$$

Next, consider the result $\alpha = s$ obtained in point **d.**

$$\alpha = s \tag{3}$$

By combining (1) or (2) with (3) (substituting s or α in one of the two expressions) you find that $k_1^* = k_2^*$! If $s \neq \alpha$ we would have two distinct expressions maximising consumption, which is not possible if we only have one maximum in steady state, as in this case.

Question: There is another (probably many) way to answer this question, can you get it?

TD2

1 Review Questions

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e. At the steady state, investments are equal to what is lost to depreciation, population growth and technological progress.

- Answer

  TRUE: At the steady state, capital per unit of effective labour $\tilde{k} = \frac{K}{AL}$ must not grow. From the results of the class

$$\Delta\tilde{k} = sf\left(\tilde{k}\right) - (\delta + n + g)\tilde{k}$$

  By setting $\Delta\tilde{k} = 0$, i.e. zero growth

$$sf\left(\tilde{k}^*\right) = (\delta + n + g)\tilde{k}^*$$

  Which exactly means that investments $\tilde{i}^* = sf\left(\tilde{k}^*\right)$ are equal to the loss in capital due to depreciation, population growth and technological progress.

f. The golden rule of savings states that in steady state, capital should be barely productive enough to compensate for depreciation, population growth and technological progress.

- Answer

  TRUE: The golden rule of savings tells us what is the optimal $\tilde{k}$ (or s, in the previous TD we found a relation between these two problems) to maximise steady state consumption $\tilde{c}^*$. To work it out we need to recall how to express consumption. We consume what we produce minus what we invest, therefore $\tilde{c}^* = (1-s)\tilde{y}^* = (1-s)f\left(\tilde{k}^*\right)$. We can exploit the expression of $sf\left(\tilde{k}^*\right)$ in steady state, which, by the result of the previous question, is the following:

$$\begin{aligned}\tilde{c}^* &= (1-s)f\left(\tilde{k}^*\right) \\ &= f\left(\tilde{k}^*\right) - sf\left(\tilde{k}^*\right) \\ &= f\left(\tilde{k}^*\right) - (\delta + n + g)\tilde{k}^*\end{aligned}$$

  What is the capital that maximises $\tilde{c}^*$? We have to solve a standard optimisation problem.

$$\max_{\tilde{k}^* \geq 0} \ \tilde{c}^* \Rightarrow \max_{\tilde{k}^* \geq 0} \ f\left(\tilde{k}^*\right) - (\delta + n + g)\tilde{k}^*$$

$$\frac{\partial \tilde{c}^*}{\partial \tilde{k}^*} = 0 \Rightarrow f'\left(\tilde{k}^*\right) - (\delta + n + g) = 0 \Rightarrow f'\left(\tilde{k}^*\right) = \delta + n + g$$

Which exactly means that the marginal productivity of capital must offset the loss due to depreciation, population growth and technological progress (remember that $f'\left(\tilde{k}^*\right)$ tells us how much production increases after an infinitesimal change in $\tilde{k}^*$, namely the marginal product of effective capital).

Question: What is the difference with what we got in the previous TD?

3 Exercise - Convergence Towards the Steady State

The aim of this exercise is to understand the role of assumptions in the Solow - Swan model. It may be tempting to read assumptions once and then forget about them, but they are of crucial importance in these and in all other theories in economics (science and reasoning in general).

Our production function for the first point is $F(K_t, L_t) = (K_t L_t)^{\frac{1}{2}} = \sqrt{K_t L_t}$. The saving rate is s, population growth rate is $\frac{\Delta L_t}{L_t} = n$ and capital depreciates at rate δ.

a. Represent graphically in the (k, y) plane the dynamics of the Solow model with population growth and no technological change.

The first step is to transform all the variables in per capita quantities. We perform this step because we are interested in the steady state of capital per worker k, and not in its absolute value K.

Question: Why? Be sure to really understand this.

- $k_t = \frac{K_t}{L_t}$
- $y_t = \frac{Y_t}{L_t}$
- $i_t = \frac{I_T}{L_t}$
- $c_t = \frac{C_t}{L_t}$

As for the production function, we perform the same changes by dividing with L_t and exploiting constant returns to scale (*CRS*).

Question: Can you prove that this production function satisfies constant returns to scale?

$$\frac{1}{L_t}F(K_t, L_t) = \left(\frac{K_t}{L_t}, \frac{\cancel{L_t}}{\cancel{L_t}}\right)^{\frac{1}{2}}$$
$$= \left(\frac{K_t}{L_t}\right)^{\frac{1}{2}}$$
$$f(k_t) = (k_t)^{\frac{1}{2}}$$
$$= \sqrt{k_t}$$

To check for the dynamics of the model we need the *law of motion of capital,* as capital is the principal state (endogenous) variable which determines what happens in the economy as time changes. The law of motion tells us how capital evolves over time. We ask ourselves the question: "*if at time t I have capital* k_t *(per capita), how much capital* k_{t+1} *do I have in the next period?*" To answer this question we need to know what is the relation between these two variables.

On the one hand, we have saved resources that we can use in the next period $sy_t = sf(k_t)$, on the other hand, capital depreciates at rate δ, and therefore we lose δk . Moreover, in this formulation of the model, there is also population growth. An increase in population decreases the amount of capital per capita. These are the two relevant factors that affects the dynamics of capital. From this intuitive reasoning you should already see that $k_{t+1} - k_t = sf(k_t) - \delta k_t - nk_t$, i.e. the difference between capital tomorrow and capital today is savings minus what is lost due to depreciation and population growth. However, we want to be rigorous and find the law of motion via rules of growth rates of products and ratios. Define $\Delta x = x_{t+1} - x_t$, then

$$\frac{\Delta k_t}{k_t} = \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} \qquad \text{Check page 17 of the lecture notes to verify that this is true}$$

We already know that $\frac{\Delta L_t}{L_t} = n$. Moreover, we know from the standard Solow model that $\Delta K_t = sF(K_t, L_t) - \delta K_t$ (why?). Therefore,

$$\frac{k_{t+1} - k_t}{k_t} = \frac{\Delta k_t}{k_t} = \frac{sF(K_t, L_t) - \delta K_t}{K_t} - n$$
$$\frac{\Delta k_t}{k_t} = \frac{\frac{1}{L_t}(sF(K_t, L_t) - \delta K_t)}{\frac{1}{L_t}K_t} - n \text{ Multiply numerator and denumerator of the fractio}$$
$$\frac{\Delta k_t}{k_t} = \frac{sf(k_t) - \delta k_t}{k_t} - n \text{ By the definition of } k_t$$

By multiplying on the right and on the left by k_t we get the final expression:

$$\Delta k_t = \underbrace{sf(k_t)}_{\text{Saved resources}} - \underbrace{\delta k_t}_{\text{Depreciation loss}} - \underbrace{nk_t}_{\text{Population growth loss}}$$
$$\Delta k_t = sf(k_t) - (\delta + n)k_t$$

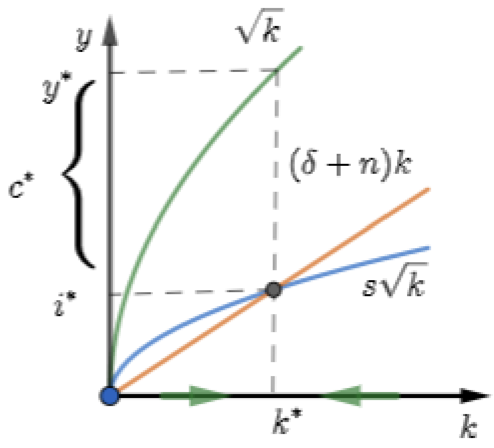
In the steady state variables do not change over time, therefore capital per capita will be stable, which means $\Delta k_t = 0$.

$$\Delta k_t = 0 \Leftrightarrow sf(k_t^*) = (\delta + n)k_t^*$$

Investment exactly offsets the loss due to depreciation and population growth.

In our case $f(k_t) = (k_t)^{\frac{1}{2}} = \sqrt{k}$, therefore the three elements of our graph are:

- $\sqrt{k}$
- $(\delta + n)k_t$
- $s\sqrt{k}$



Graph1: Dynamics of the Solow Model with Population Growth.

Remark: Always, always, always put labels on axes when you draw graphs!

As always we have that $y^* = f(k^*)$, $i^* = sf(k^*)$ and $c^* = f(k^*) - sf(k^*) = y^* - i^*$, as shown in the graph.

b. Same question for the AK production function $F(K_t, L_t) = AK_t$, assuming $sA > \delta + n$. Does k^* exist in this case?

For this point, I offer a different path to reach the solution compared to what you will receive from the professor. I think my way is more in line with the standard method, but you choose which one you prefer.

To answer this question we proceed as we did in the previous point, but of course, we have to take into account the different production function and the technological change. First, let's express F as a function of capital per capita. We perform the same computation as before.

$$\frac{1}{L_t}F(K_t, L_t) = \left(A\frac{K_t}{L_t}\right)$$
$$f(k_t) = Ak_t$$

Notice that, contrary to what you see in the lecture notes, the exercise asks you to draw the graph in the space (k, y) and not $\left(\tilde{k}, \tilde{y}\right)$, that's why I do not divide by units of effective labour AL.

Question: What if we had $\tilde{k} = \frac{K_t}{AL}$? **Answer:** Due growth rates rules:

$$\begin{aligned}\frac{\Delta \tilde{k}}{\tilde{k}} &= \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} - \frac{\Delta A}{A} \\ &= \frac{sF(K_t, L_t) - \delta K_t}{K_t} - n - g \\ \Delta \tilde{k} &= sf\left(\tilde{k}\right) - (\delta + n + g)\tilde{k}\end{aligned}$$

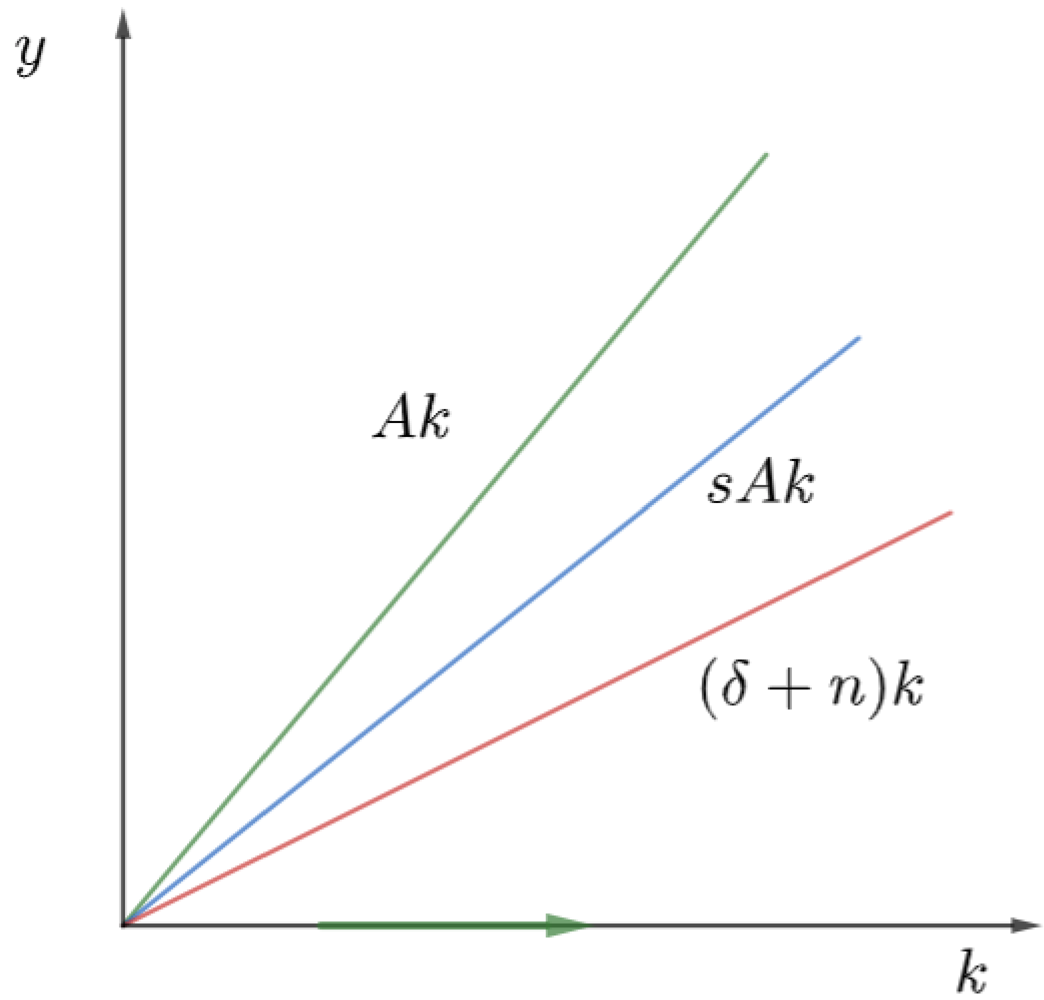
However, we are still in the (k, y) space, so we are interested in the law of motion of capital per capita k, which is the same we had in the previous point (except for the different production function). We have the following:

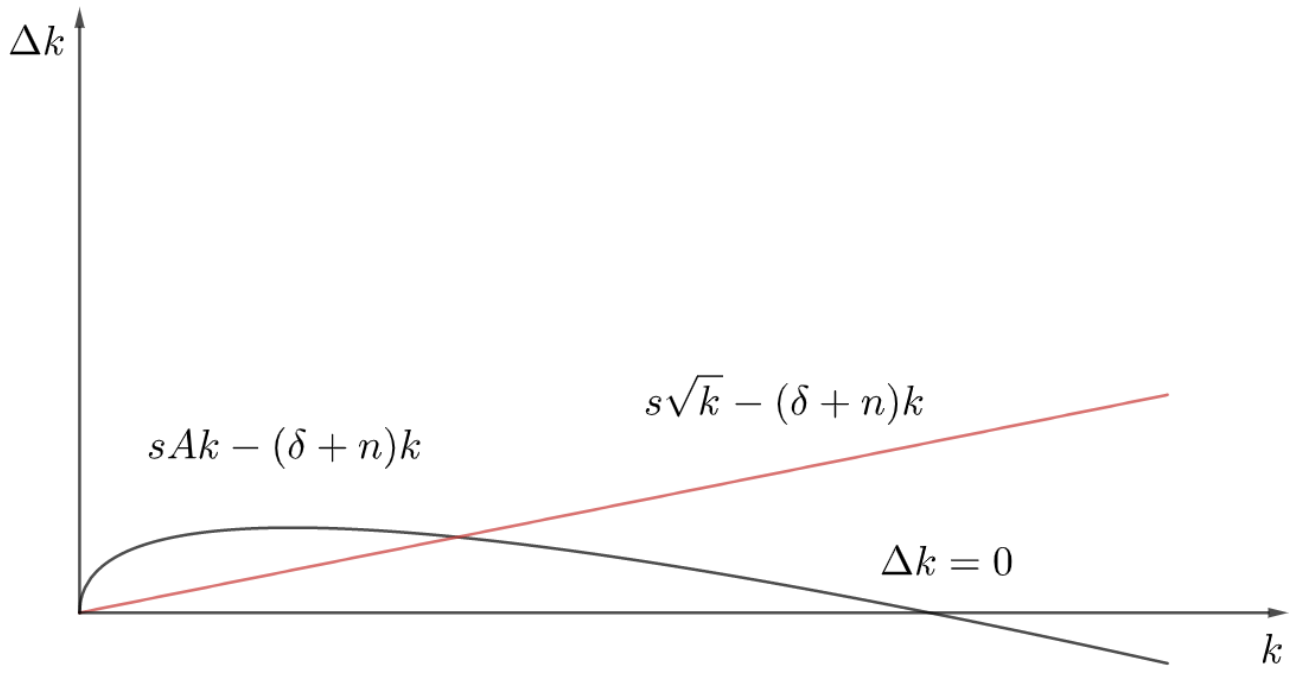
$$\begin{aligned}\Delta k_t &= sf(k_t) - (\delta + n)k_t \\ &= sAk_t - (\delta + n)k_t\end{aligned}$$

So, does k^* exists in this case? The steady state condition is always the same:

$$\Delta k_t = 0 \Leftrightarrow sf(k_t^*) = (\delta + n)k_t^* \Leftrightarrow sA \cancel{k_t^*} = (\delta + n) \cancel{k_t^*} \Leftrightarrow sA = (\delta + n)$$

However, in this model, this can never be true as we have $sA > \delta + n$! So the answer is no, there exists no steady state capital k^* (notice however that $k^* = 0$ is a solution, in fact, the condition $sAk_t^* = (\delta + n)k_t^*$ is respected in this case). The reason is also apparent from the graph. Inspired by a question of one of your classmates I also plotted a graph of the two economies we just studied in the $(k, \Delta k)$ space.





Graph 2 and 3: Dynamics of the Solow model with technological change and linear production function. *Question*: Can you guess what point the intersection between the curve and the x-axis is?

The problem here is that the saving rate and the technological change outset the decrease of capital per capita due to depreciation and population growth. This is due to the fact that the coefficient of the (linear) savings function sA is greater than the coefficient of the depreciation $\delta + n$. Therefore, the increase in capital will always be greater than its loss. Its growth will never stop, it will continue to increase in every time t. This example shows why assumptions are a fundamental ingredient of the model and not something we use just for convenience and that we can forget by putting them below the carpet. We will discuss this in the following point.

Question: Do you know what would happen if $sA < \delta + n$?

Question: Do you think the results would have been different if we checked capital per units of effective labour $\left(\tilde{k} = \frac{K_t}{AL_t}\right)$?

c. Make a list of the properties that the AK function does not satisfy with regards to the Solow model. Which one explains the previous result?

In the Solow model, there are three assumptions on the production function and three extra assumptions that are dubbed *Inada Conditions*.

1. $F(K) > 0$ if $K > 0$. This condition ensures that our production is positive if we use a positive amount of capital;
2. $\frac{\partial F(K)}{\partial K} > 0$. This condition tells us that we have a *positive marginal product*, that is, increasing capital always increases production;
3. $\frac{\partial^2 F(K)}{\partial K^2} < 0$. This condition is a crucial one in this exercise. It ensures that the marginal product is decreasing in K. This means that the n^{th+1} unit of capital K will increase the production less than the n^{th} one. We will show that this does not hold in the previous point.

Then we have the *Inada Conditions.*

1. $F(0) = 0$. This imposes that you can not have a positive production by employing zero capital;
2. $\lim_{K \to 0} \frac{\partial F(K)}{\partial K} = \infty$. This assumption tells us that a little bit of capital is infinitely productive, as we go from 0 production to positive production;
3. $\lim_{K \to \infty} \frac{\partial F(K)}{\partial K} = 0$. This assumption ensures that employing an infinite amount of capital is not convenient, as the marginal product will eventually reach 0 so that using capital will not be productive at all and therefore will be wasted.

Let's check that $F(K_t, L_t) = AK_t$ does not satisfy assumption 3 and Inada condition 3 . and hence leads to no positive steady state level of capital per capita. We have

$$\frac{\partial F(K)}{\partial K} = A > 0$$

$$\frac{\partial^2 F(K)}{\partial K^2} = 0 \geq 0$$

$$\lim_{K\to\infty} \frac{\partial F(K)}{\partial K} = A \neq 0$$

Therefore, as we noticed before, capital is too productive and its increase due to production always offset its loss due to depreciation and population growth.

Question: Check that the production function in the first part of the exercise indeed satisfies all these assumptions.

Question: Are you sure that $F(K_t, L_t) = AK_t$ satisfies all other assumptions except 3 and Inada condition 3?

TD3

1 Review Questions

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b. On a balanced growth path, all variables grow at the same rate.

- *Answer*

  False: Recall the definition of Balanced growth path at page 19 on you lecture notes:

  > *Definition: A balanced growth path is a trajectory such that all variables grow at a constant rate.*

  Translated in mathematical terms, we have that all the variables x_i in our model must have $g_{x_i} = k_i$, where k_i is a constant. However, it is not specified that all k_i must be equal! Each variable can grow at its own, constant rate. We have an example in the exercise below, where different variables of interest grows at different rates.

c. The Solow model needs to assume technological change to check the stylised Kaldor facts of growth.

- *Answer*

  True: Consider as an example Kaldor fact 1:

  > *Kaldor fact 1: Labour productivity has grown at a sustained rate.*

  If we do not have technology, labour productivity does not grow in the steady state. In fact, if we do not have technology and we are in a steady state then $g_k = g_K - g_L = 0$, since $y = f(k)$, if k does not grow then also y will not grow. Since $y = \frac{Y}{L}$, it is a measure of labour productivity (i.e. production per capita). You can check this on page 19 of your lecture notes, but the next exercises constitute a clear example of why this is true. Introducing a technological shift which grows at rate g makes growth positive.

  Question: Try to argue the same thing by considering Kaldor fact 2 about capital per worker.

d. The Solow model predicts convergence of all economies in the world to the same GDP per capita.

- *Answer*

  False: The Solow model can be interpreted as a machine that takes as an input exogenous parameter $(n, \delta, etc\ldots)$ and tells you what happens to the economy. To a different set of exogeneous parameters we obtain a different prediction. Consider as an example the economies at points a. and b. of exercise 3 of this TD(1), the growth predictions are completely different.

4 Problem - The Solow Model with Natural Resources

The aim of this exercise is to get you used with growth rates calculations. It is in some sense less interesting from an intuitive point of view, but we will be able to link it to exercise 2 in TD2.

We have quite a lot of data. The production here is affected by three variables, capital K_t, labour L_t and a natural resource Z_t. We also have capital augmenting technology A_t. The function is the following

$$Y_t = (A_t K_t)^{\alpha} (L_t)^{1-\alpha-\beta} Z_t^{\beta}$$

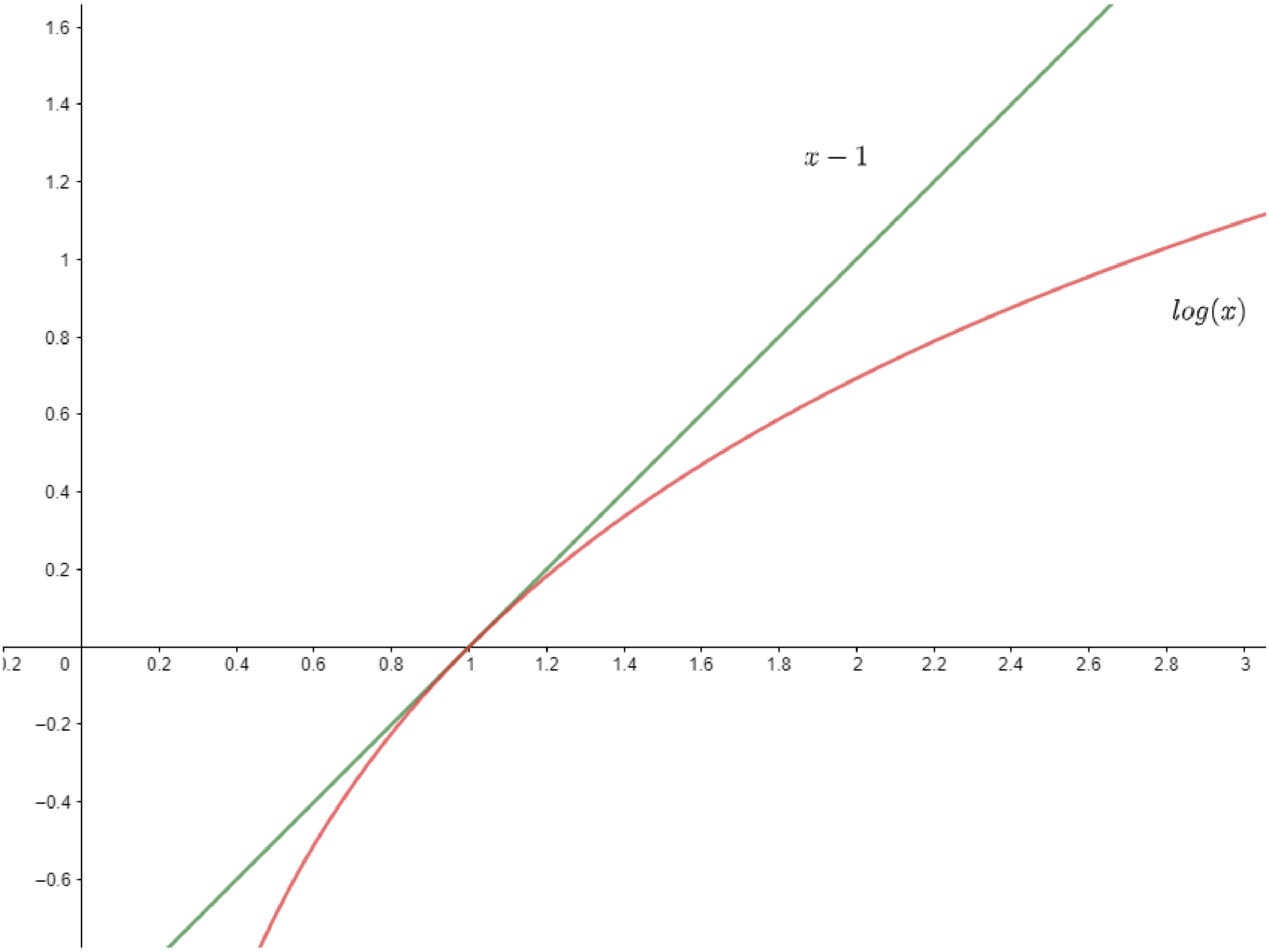
The law of motion of capital is the standard one $\Delta K_t = K_{t+1} - K_t = (1-\delta)K_t + I_t$ where $I_t = sY_t$. Technology, grows at an exogenously fixed rate $A_{t+1} = (1+\gamma)A_t$. Also the stock of natural resources grows at an exogenous fixed rate $Z_{t+1} = (1+\varepsilon)Z_t$. Labour also grows, as we already saw $L_{t+1} = (1+n)L_t$.

Throughout the problem we will use the following useful approximation:

$$log\left(\frac{X_{t+1}}{X_t}\right) \approx \frac{X_{t+1}}{X_t} - 1$$

Remember that $\frac{X_{t+1}}{X_t} - 1 = \frac{X_{t+1}-X_t}{X_t} = \frac{\Delta X_t}{X_t}$, the growth rate.

Question: Check the graph below. Do you think this approximation always works?



Graph1: Logarithmic approximation.

a. In this problem use g_x to denote the growth rate of the variable x (for example $g_y = log\left(\frac{Y_{t+1}}{Y_t}\right)$). From the definitions, write g_A, g_L and g_Z.

Let's use the definition and the approximation we are given. We start from g_A.

$$\begin{aligned}(1+\gamma)A_t &= A_{t+1} \\ (1+\gamma) &= \frac{A_{t+1}}{A_t} \\ \gamma &= \frac{A_{t+1}}{A_t} - 1 \approx log\left(\frac{A_{t+1}}{A_t}\right) \\ \gamma &= g_A\end{aligned}$$

We can perform the same calculations to see that $g_L = n$ and $g_Z = \varepsilon$. *Question*: Try to find g_L and g_Z as an exercise.

b. Compute g_Y in terms of $\alpha, \beta, g_A, g_K, g_Z$, and g_L.

This seems like a daunting task, so let's divide this computation by steps.

First, we must identify the variable of which we want to compute the growth rate. In this case we have from the text $Y_t = (A_t K_t)^\alpha (L_t)^{1-\alpha-\beta} Z_t^\beta$.

Second, we use the explicit expression of growth rates to understand how its growth rate is composed. Since we have that $g_Y = log\left(\frac{Y_{t+1}}{Y_t}\right)$, we first have to compute $\left(\frac{Y_{t+1}}{Y_t}\right)$.

$$\begin{aligned}\left(\frac{Y_{t+1}}{Y_t}\right) &= \frac{(A_{t+1}K_{t+1})^\alpha (L_{t+1})^{1-\alpha-\beta} Z_{t+1}^\beta}{(A_t K_t)^\alpha (L_t)^{1-\alpha-\beta} Z_t^\beta} \\ &= \left(\frac{A_{t+1}}{A_t}\frac{K_{t+1}}{K_t}\right)^\alpha \left(\frac{L_{t+1}}{L_t}\right)^{1-\alpha-\beta} \left(\frac{Z_{t+1}}{Z_t}\right)^\beta\end{aligned}$$

Third, we take logs, so that we have a direct expression for g_Y.

Remark: Remember that $log\,(a \cdot b) = log\,(a) + log\,(b)$ and $log(a)^\alpha = \alpha \cdot log\,(a)$.

$$\begin{aligned}log\left(\frac{Y_{t+1}}{Y_t}\right) &= log\left[\left(\frac{A_{t+1}}{A_t}\frac{K_{t+1}}{K_t}\right)^\alpha \left(\frac{L_{t+1}}{L_t}\right)^{1-\alpha-\beta} \left(\frac{Z_{t+1}}{Z_t}\right)^\beta\right] \\ &= \alpha\left(log\left(\frac{A_{t+1}}{A_t}\right) + log\left(\frac{K_{t+1}}{K_t}\right)\right) + (1-\alpha-\beta)\left(log\left(\frac{L_{t+1}}{L_t}\right)\right) + \beta\left(log\left(\frac{Z_{t+1}}{Z_t}\right)\right) \\ &= \alpha\,(g_A + g_K) + (1-\alpha-\beta)\,(g_L) + \beta\,(g_Z) \\ g_Y &= \alpha\,(\gamma + g_K) + (1-\alpha-\beta)\,(n) + \beta\,(\varepsilon)\end{aligned}$$

We have exactly g_Y in terms of $\alpha, \beta, g_A, g_K, g_Z$, and g_L.

c. Compute g_K in terms of δ, s and $\frac{Y_t}{K_t}$.

Exactly as before, we exploit the definition of growth rate and what we know about K_t. The law of motion of capital is always the same.

$$K_{t+1} - K_t = \Delta K_t = sF(A_t, K_t, L_t, Z_t) - \delta K_t$$

We elaborate a little bit on this expression to put it in a form that is convenient to us. First, we divide by K_t to explicitly have the growth rate.

$$\begin{aligned}\frac{K_{t+1} - K_t}{K_t} &= \frac{sF(A_t, K_t, L_t, Z_t) - \delta K_t}{K_t} \\ \frac{K_{t+1}}{K_t} - 1 &= s\frac{Y_t}{K_t} - \delta \\ log\left(\frac{K_{t+1}}{K_t}\right) &\approx s\frac{Y_t}{K_t} - \delta \quad \text{(By the approximation given in the text)} \\ g_K &\approx s\frac{Y_t}{K_t} - \delta\end{aligned}$$

We managed to find an expression of g_K in terms of δ, s and $\frac{Y_t}{K_t}$.

d. Argue why, along a balanced growth path, $\frac{Y_t}{K_t}$ must be constant. Then argue why $g_Y = g_K$.

Recall the definition of a balance growth path: all the variables must grow at a constant rate! This means, in order, that K_t must grow at a constant rate, that g_K must be equal to a constant, and that $s\frac{Y_t}{K_t} - \delta$ must be constant. We know that s and δ are indeed constant, but, if we are not on a balance growth path $\frac{Y_t}{K_t}$ evolves with time. Therefore, $\frac{Y_t}{K_t}$ must not change for g_K to be constant, so that K_t grows at a constant rate.

As for the second question, the answer is only one step ahead of the previous reasoning. In order for the ratio $\frac{Y_t}{K_t}$ to be constant, the two variables must grow at the same rate in each time t. If, as an example, K_t grows quicker than Y_t, the ratio will not be constant in time, therefore $g_Y = g_K$. More precisely, if $\frac{Y_t}{K_t}$ grows at a constant rate, it means that $g_{\frac{Y_t}{K_t}} = 0$. By exploiting the rules of growth rates:

$$g_{\frac{Y_t}{K_t}} = 0 \Leftrightarrow g_Y - g_K = 0 \Leftrightarrow g_Y = g_K$$

Which is what we wanted to prove.

e. Using your answers to earlier parts of the problem, solve for g_Y in terms of $\alpha, \beta, \gamma, \varepsilon$ and n (from now on I omit the index t for simplicity).

From point **b.** we have that $g_Y = \alpha(\gamma + g_K) + (1 - \alpha - \beta)(n) + \beta(\varepsilon)$, while form point **d.** we know that along a balanced growth path $g_Y = g_K$. By substituting the second condition into the first on we obtain:

$$\begin{aligned} g_Y &= \alpha(\gamma + g_K) + (1 - \alpha - \beta)(n) + \beta(\varepsilon) \\ &= \alpha\gamma + \alpha g_Y + (1 - \alpha - \beta)(n) + \beta(\varepsilon) \\ g_Y(1-\alpha) &= \alpha\gamma + (1 - \alpha - \beta)(n) + \beta(\varepsilon) \\ g_Y &= \frac{\alpha\gamma + (1 - \alpha - \beta)(n) + \beta(\varepsilon)}{1-\alpha} \end{aligned}$$

Which gives us $g_Y$in terms of $\alpha, \beta, \gamma, \varepsilon$ and n.

f. What is the condition for $g_{\frac{Y}{L}}$ to be positive along a balanced growth path? Interpret.

To answer this question we have first to compute the quantity of interest. The rule is always the same:

$$
\begin{aligned}
g_{\frac{Y}{L}} &= g_Y - g_L \\
&= \frac{\alpha\gamma + (1-\alpha-\beta)(n) + \beta(\varepsilon)}{1-\alpha} - n \\
&= \frac{\alpha\gamma + (1-\alpha-\beta)(n) + \beta(\varepsilon) - n(1-\alpha)}{1-\alpha} \\
&= \frac{\alpha\gamma + n - \alpha n - \beta n + \beta(\varepsilon) - n + \alpha n}{1-\alpha} \\
g_{\frac{Y}{L}} &= \frac{\alpha\gamma - \beta n + \beta(\varepsilon)}{1-\alpha}
\end{aligned}
$$

Now we are ready to evaluate when this expression is positive. First, we know that the denominator is always positive, as $\alpha < 1$. Therefore, the whole fraction is positive when the numerator is positive.

$$
g_{\frac{Y}{L}} > 0 \Leftrightarrow \alpha\gamma - \beta n + \beta\epsilon > 0 \Leftrightarrow \alpha\gamma + \beta\epsilon > \beta n
$$

Before interpreting this result, we must understand what $g_{\frac{Y}{L}}$ indicates. It is the growth rate of what we usually denote y, production in per capita terms. So, asking when $g_{\frac{Y}{L}}$ is positive is the same as asking: "when does production in per capita terms has a positive growth rate?". Hopefully this interpretation of the question helps us understand this condition. There are three factors that affect consumption per capita.

$$
\underbrace{\alpha\gamma}_{\text{Technological growth}} \quad + \quad \underbrace{\beta\varepsilon}_{\text{Natural Resource Growth}} \quad > \quad \underbrace{\beta n}_{\text{Population growth}}
$$

The first two increases product per capita, while the third one decreases it. Therefore, growth will be positive when the sum of the first two is higher than the third.

Question: Can you guess what is the role of α and β exactly?

2 Solow-Swan with Non-renewable Resources (from TD2!)

This problem is tightly related to the previous one but it has less computations and more intuition. Its focus is to study the employment of renewable and non-renewable resources and its sustainability.

We have the same production function and growth rates as before.

$$
Y_t = (A_t K_t)^{\alpha} (L_t)^{1-\alpha-\beta} Z_t^{\beta}
$$

However, here we specify how Z is composed. We can split natural resources in renewable R and non-renewable N. The rate of exploitation of N is r, therefore we have that $Z = R + rN$.

a. Assume that initially, the economy is in a balanced growth path (BGP) where the stock of renewable resources is stable and where there are no non-renewable resources at all. What is the growth rate of y? Interpret in what conditions we get a positive rate of growth for y. Knowing what we can anticipate about the rate of

technology and population growth in the 21^{st} century, should we expect y to grow or not in the coming decades ?

We already have the growth rate of $y = \frac{Y}{L}$ on a balanced growth path from the previous exercise.

$$g_y = g_{\frac{Y}{L}} = \frac{\alpha\gamma - \beta n + \beta(\epsilon)}{1-\alpha}$$

However, in this case there are no non-renewable resources $N = 0$ and the stock of renewable resources R is stable, which implies that it is not growing. Since $Z = R + rN$ and $N = 0$ we have that here $Z = R$. The growth rate of Z was ε in the previous exercise, but since here R does not grow, Z does not grow either, as it is composed by R only. This translates into $\varepsilon = 0$. The new growth rate is therefore:

$$g_y = \frac{\alpha\gamma - \beta n}{1-\alpha}$$

The evaluation of its sign is the same as before. In particular, we know that $1-\alpha$ is always positive, therefore the sign of the numerator is the significant one. The whole fraction is positive when the numerator is positive:

$$g_y > 0 \Leftrightarrow \alpha\gamma > \beta n$$

We have 4 factors that affect this inequality:

1. α captures the relative importance of capital in the production function. Intuitively, if capital is relatively more important there are more chances that per capita growth is positive, as it is directly affected by technological progress;
2. γ is the rate of growth of technology. Of course, the more technology improves the more likely is that growth per capita increases;
3. β is α counterpart for natural resources, it measures its relative importance in the production. Since these do not grow, if they are less important than growth will be positive despite the fact that the stock is fixed;
4. n represents the growth rate of population. Intuitively, if population increases the growth per capita decreases, as there are more mouths to feed.

Since we know that n is quite low while γ is high, if the premises of this model are true then we would be sure to enjoy positive growth in the future.

TD4

1 Review Questions

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a. In an ecosystem, the natural growth of a renewable resource is an increasing function of the amount of this resource.

- *Answer*

  False: As an example, in class, you considered a logistic growth, captured by the equation $\tau(S_t) = rS_t\left(1 - \frac{S_t}{K}\right)$. As you can see, when $S_t \to K$ then $\tau(S_t) \to 0$. Therefore, it is not true that if S_t increases then its growth also increases.

b. An improvement in extractive technology always increases fish production if fishing is free.

- *Answer*

  False: In our model the total production of fish when there is free entry is given by the following:

$$H_F = B_F \alpha S_F = \left(1 - \frac{c}{p\alpha K}\right)\frac{r}{\alpha}\frac{c}{p}$$

  As you can see, we have an α at the denominator with a minus sign (positive effect on H_F), but we also have an α at the denominator with a plus sign (negative effect on H_F). Hence, the total effect is ambiguous.

2 Solow-Swan with Non-renewable Resources

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b. Assume that at time t (when the economy was previously on the balanced growth path), a new source of non-renewable resource of size N_t is discovered. Each ensuing period, $r\%$ of the resource stock is used in production, such that its stock goes progressively to zero in the long run. If $R = 1$, $N_t = 20$ and $r = 0.1$, what is the growth rate of the supply of resource Z before t? Right after t? In the very long run?

Before answering this question we have to compute the growth rate of Z_t. In this case the expression of interest is a sum, therefore we can not use the rules of product and ratios of growth rate. The idea is to subtract Z_{t-1} to Z_t in order to find the Δ. The calculations to attain the growth rate are the following (I omit t in the computations):

$$\begin{aligned}
Z_t &= R_t + rN_t \\
Z_t - Z_{t-1} &= R_t - R_{t-1} + r(N_t - N_{t-1}) \\
\Delta Z &= \Delta R + r\Delta N \\
\frac{\Delta Z}{Z} &= \frac{R}{R}\frac{\Delta R}{Z} + r\frac{N}{N}\frac{\Delta N}{Z} \\
\frac{\Delta Z}{Z} &= \frac{R}{Z}\frac{\Delta R}{R} + r\frac{N}{Z}\frac{\Delta N}{N} \\
g_Z &= \frac{R}{Z}g_R + r\frac{N}{Z}g_N \\
g_{Z,t} &= \frac{R}{R+rN_t}g_R + r\frac{N_t}{R+rN_t}g_{N,t}
\end{aligned}$$

By substituting the numbers we have we obtain the growth rate when the new non-renewable resource is discovered, at time t. Remember that R does not grow ($g_R = 0$) and that Z has a negative growth of -0.1 We have:

$$\begin{aligned}
g_{Z,t} &= \frac{R}{R+rN}g_R + r\frac{N}{R+rN}g_N \\
g_{Z,t} &= \frac{1}{1+(0.1)(20)}(0) + \frac{(0.1)(20)}{1+(0.1)(20)}(-0.1) \\
g_{Z,t} &= \frac{(0.1)(20)}{1+(0.1)(20)}(-0.1) = -6.\bar{6}\% = -\frac{1}{15}
\end{aligned}$$

As for $g_{Z,\tau}$ for $\tau < t$, we have that $g_{Z,\tau} = 0$, as $N_t = 0$ and R does not grow, exactly as we had in the previous point. Instead, when $\tau \to \infty$ the growth rate also goes to zero. This is due to the fact that Z has a negative growth, and therefore after it is completely exploited it will not grow (negatively) anymore.

For the last points, the professor run an analysis with particular values of parameters $n = 0.01$, $g = 0.015$, $\delta = 0.02$, $s = 0.3$, $\alpha = 0.3$, $\beta = 0.2$, $R = 1$, $N_t = 20$, $r = 0.1$. Finally you see the model at work! The graphs in the text of the exercise are the plot of three time series: $\frac{Y}{K}$, $\frac{Y}{L}$ and $\frac{K}{L}$. The continuous line describe an economy in a balanced growth path with no non-renewable resources, while the dashed line depicts a scenario where non-renewable resources are discovered at time t. The main of the following points is to connect the graphs to ratios.

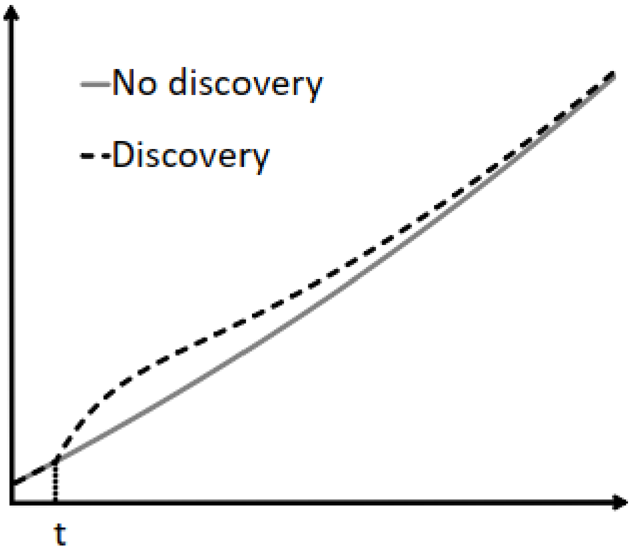
c. Which graph is $\frac{K}{L}$? Explain in words what happens at time t and in the ensuing periods.

Let's try to find a general way of answering these kind of questions. The variables involved are K, L and Y. The question is: how are these variables affected by an increase in N? To answer we need to know the dependencies that all these variables have with N. As an example, L is only determined by its growth, we start from L_0 and then we get L_1 based on how big n is. Therefore, L is not directly affected by N. The same holds for K, its growth is given by the growth rate g_K, and not directly by N. Hence, K also is not directly affected by N. Instead,

$Y_t = (A_t K_t)^\alpha (L_t)^{1-\alpha-\beta} Z_t^\beta$ where $Z_t = R + rN_t$. A jump in N causes an immediate impact in Y. This analysis offers a first insight to answer this question as immediate impacts are visible in the graph as "jumps". We must consider the three ratios $\frac{K}{L}$, $\frac{Y}{K}$ and $\frac{Y}{L}$. Of these three, the only ratio that does not "jump" is $\frac{K}{L}$. Hence, we are sure that the right graph is *C*, as there is a smooth evolution of the dashed line at time t.

To understand what happens here recall that $g_K = s\frac{Y_t}{K_t} - \delta$. Since Y increases, as we elaborated before, the growth rate of K increases, so K increases more then what it would without N. However, N will go to zero slowly, which means that the accumulation of capital K slowly go back to its original path.

C

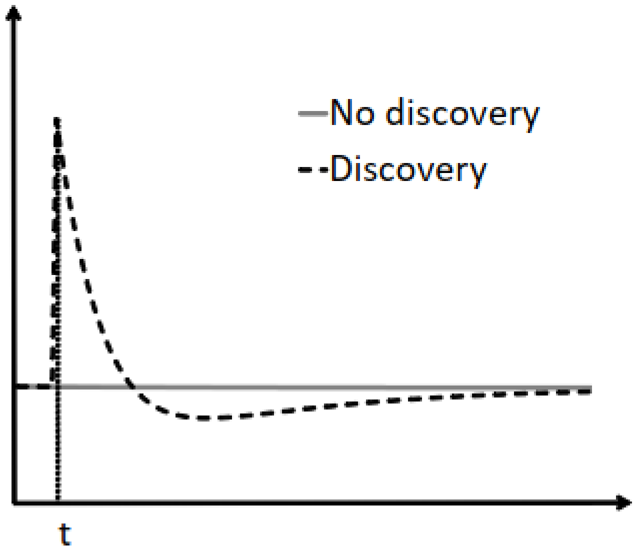


d. Which graph is $\frac{Y}{K}$? Explain in words what happens at time t and in the ensuing periods.

First step done, now we have to distinguish between $\frac{Y}{L}$ and $\frac{Y}{K}$. The difference between the two graphs we are left with is that one of the the balanced growth path is constant (flat horizontal line). Therefore, we have to answer the question: which ration between $\frac{Y}{K}$ and $\frac{Y}{L}$ should be fixed without the increase in N? Well, we now that the growth rate of L is exogenously given and it is $n > 0$, so it is impossible that L will be fixed. Also Y and K will grow, but at which rate? We know from the previous TD that $g_{\frac{Y_t}{K_t}} = 0 \Leftrightarrow g_Y - g_K = 0 \Leftrightarrow g_Y = g_K$, therefore $\frac{Y}{K}$ is constant. Hence, it is represented by graph *B*.

Here two things happen. First, as we saw before, Y jumps. Instead, K does not jumps immediately, but smoothly increases. Therefore, at time t , the ratio $\frac{Y}{K}$ will steadily increase ("jump"). However, the push of Y is only big at time t, after that the growth rate of Y will slowly return normal. Instead, the growth rate of K will increase, and the increase of K will offset the increase of Y. This explains why the dotted line goes below the horizontal line. After some time also the growth of K returns normal, and the dotted line returns on the old balanced growth path.

B

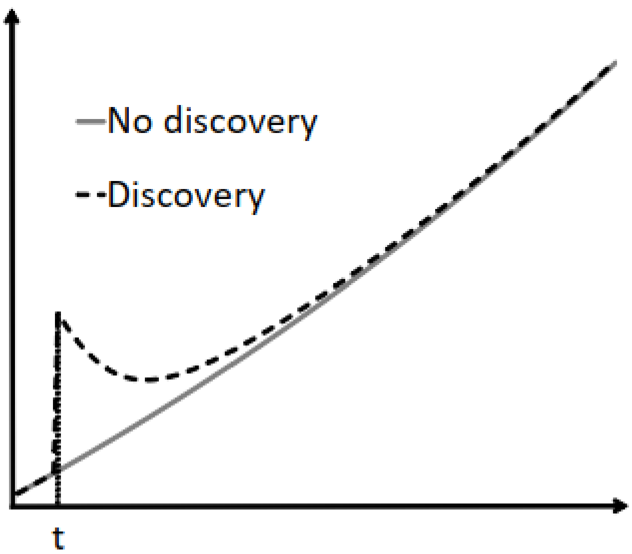


e. Which graph is $\frac{Y}{L}$? Explain in words what happens at time t and in the ensuing periods.

We are only left with one ratio and graph A, so the answer is easy here, but we must understand also what is going on.

The jump is always given by the steady increase of Y, as in the previous points. However, L will be always increasing with the same rate, in contrast to K in the previous graph. Nevertheless, the ratio immediately starts to decrease after t, slowly reaching the old balanced growth path. This is due to the fact that the stock of natural resources is depleted with a rate way higher $\left(-6.\bar{6}\%\right)$ than the rate at which productivity increases (1.5%).

A



> "We are in the beginning of mass extinction, and all you can talk about is money and fairy tales of eternal economic growth"- Greta Thunberg at the United Nations Climate Action Summit, September 23, 2019

f. There is little doubt that a mass extinction is going on. However, this widespread idea of sustained economic growth being a myth is open to debate. Expanding on the model, can we comment on it.

This is more of a philosophical question, so fell free to answer whatever you think that is relevant and consistent with the model (it is true that many answer could be consistent with what we have here, but there are also many things that are not!). What this little exercise can tell us is that there is no need of non-renewable resources for an economy to grow, as long as other rates $(n, g \ldots)$ are positive. However, a sudden increase in N also brings to a jump in Y. Maybe this model could also suggest that we need a fixed amount of renewable resources R, even if it does not grow. Therefore, according to this model, it might be not a very good idea to destroy forests 🙄.

3 The Dynamics of a Fish Population with Threshold

One of the problems that the fishing model has is that the only circumstance in which there is an extinction of fishes (or natural resources in general) is when the starting stock is equal to 0. Of course, this is counterfactual with reality as we go from a state in which there is a positive amount of resources to a state in which they are extinct. The aim of this exercise is to augment to model by assuming that when the stock of fishes goes below a threshold T then it is destined to converge to 0. I think it is an interesting exercise, it helps interpreting some real world facts.

a. Find the values of $S(t) > 0$ for which the fish stock does not grow naturally.

As always, we first need to understand what the question is asking. When does the fish stock grows? When its growth rate is different then 0. If the growth rate is equal to zero then the stock will not grow. The question is asking ask to determine for which values of $S(t)$ the growth rate is equal to 0. The growth rate is:

$$N(S(t)) = r(S(t) - T)\left(1 - \frac{S(t)}{K}\right)$$

Since $r > 0$ and the expression is a product, we have that $N(S(t)) = 0$ when one of the two element of the product is zero.

$$N(S(t)) = 0 \Leftrightarrow \begin{cases} S(t) - T = 0 \\ 1 - \frac{S(t)}{K} = 0 \end{cases} \Leftrightarrow \begin{cases} S(t) = T \\ S(t) = K \end{cases}$$

Hence, fishes will not grow when their stock is exactly equal to their maximum capacity K and when the stock is equal to the minimum treshold T. Notice that this is clear also from the plot of the growth rate.

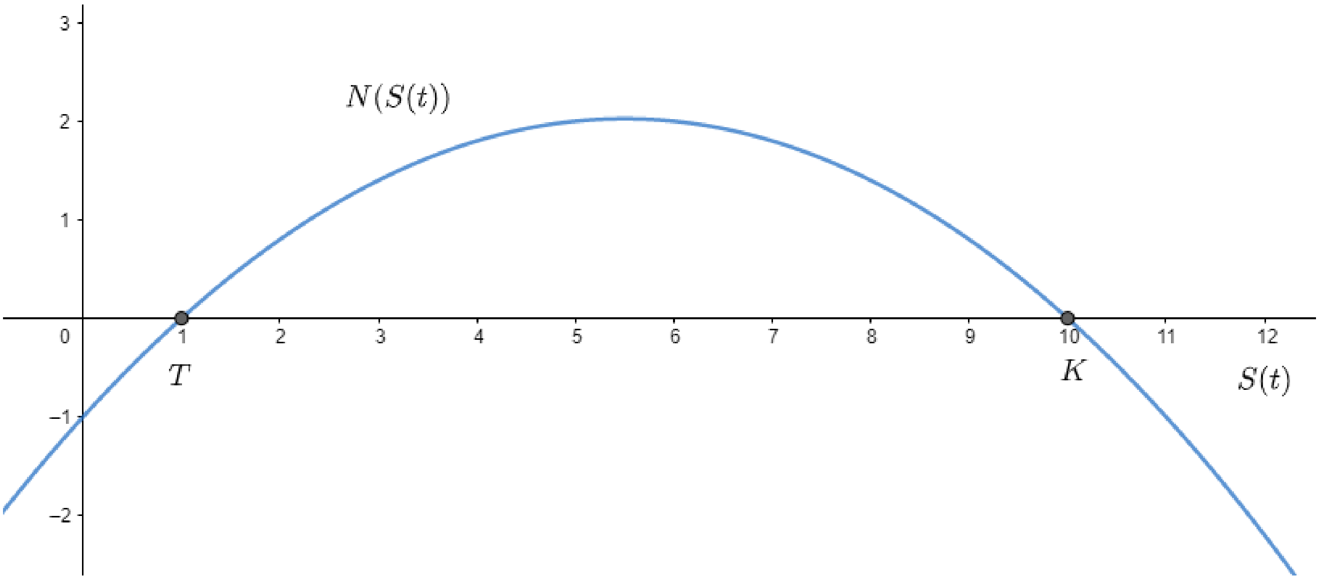


Figure 1: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$.

b. Of these values, which are stable, which are not?

First of all, what does stable mean in this context? When speaking about steady states (growth rates equal to 0), we say that a steady state is stable if a small perturbation of the system from a steady steady returns to the previous point autonomously. In this case, a steady state is stable if by slightly increasing or decreasing the stock of fishes from $S(t) = T$ or $S(t) = K$ we then return to the previous steady state or the system evolves in a different direction.

Checking for stability seems a daunting task, but if you have the graph it becomes easier. Here I will show you to check for stability with both a graphical and a mathematical technique. Let's start from the graphical one. Consider the steady state $S(t) = T$, what happens if we move slightly on the right (e.g. $S(t) = T + \varepsilon$) ? You see that $N(T + \varepsilon)$ is positive. Hence, the stock will continue to grow and will become significantly different with respect to $S(T) = T$. In the same way, if we perturb the stock in the other direction ($S(T) = T - \varepsilon$) we can see that $N(T - \varepsilon)$ is negative, the stock will become lower and lower. This steady state is not stable. If we perturb it slightly the system will not return to its original point. Now consider the second steady state, when $S(t) = K$. If we perturb it by moving slightly on the right ($S(K) = K + \varepsilon$) we see that $N(K + \varepsilon))$ is negative, therefore the stock of fish will decrease until it returns to its stable value $S(t) = K$. The same happens when you perturb the stock in the other direction, we have that $N(K - \varepsilon))$ is positive, which will make the stock increase until it is stable in $S(t) = K$. We concluded that $S(t) = T$ is not stable while $S(t) = K$ is stable. The graph below captures this reasoning pattern.

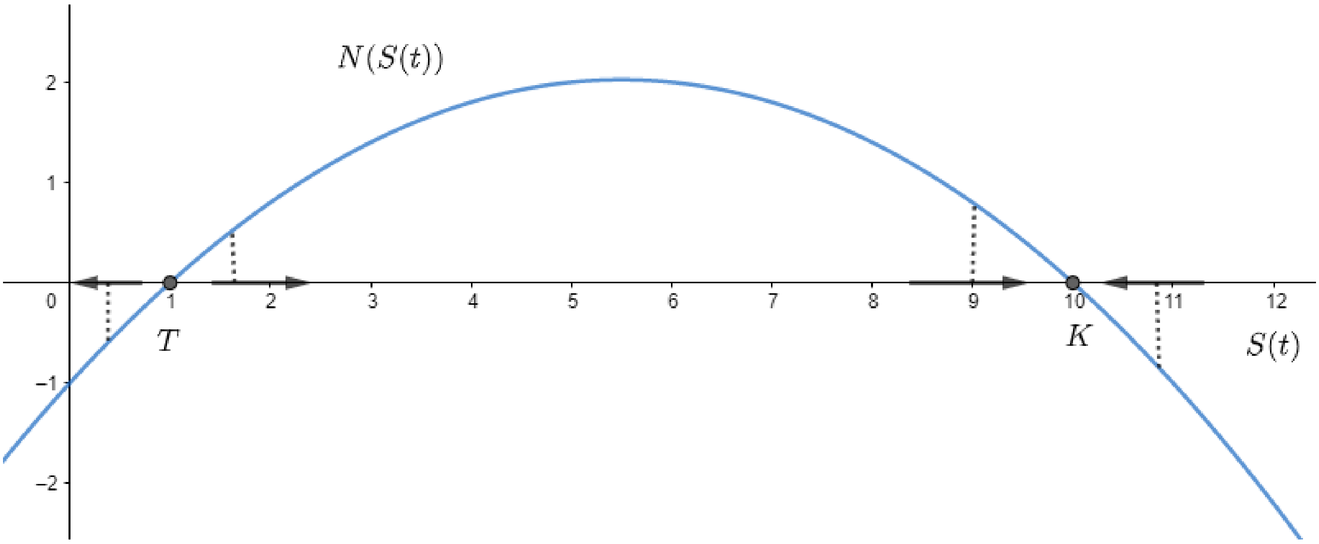


Figure 2: Stability of steady states.

If you don't like this graphical reasoning, there is also the math way. This amounts to taking the derivative of $N(S(t))$ with respect to $S(t)$, which measures the change in growth by a change in stock of fish. By evaluating the derivative in the two steady states we can check its sign. If the sign of the derivative is positive, this means that a positive variation will be magnified even more, and therefore the steady state is not stable. If the sign is negative, it means that after a positive variation the stock will decrease and return to its original value. This would mean that the steady state is stable. Let's start taking the derivative. It might look a little bit scary to do the the derivative with respect to $S(t)$, but you just have to consider the entire expression as a single variable and derive it with the rules you know (you can look at TD1 for an explanation on how to derive products).

$$\begin{aligned}\frac{\partial N(S(t))}{\partial S(t)} &= r\left[\left(1-\frac{S(t)}{K}\right)+(S(t)-T)\left(-\frac{1}{K}\right)\right]\\ &= r\left[1-\frac{S(t)}{K}-\frac{S(t)}{K}+\frac{T}{K}\right]\\ N'(S(t)) &= r\left[1-\frac{2S(t)}{K}+\frac{T}{K}\right]\end{aligned}$$

We can now evaluate the derivative in the two points of interest. Recall that $r > 0$.

$$\begin{aligned}N'(T) &= r\left[1-\frac{2S(t)}{K}+\frac{T}{K}\right]\\ &= r\left[1-\frac{2T}{K}+\frac{T}{K}\right]\\ &= r\underbrace{\left[1-\underbrace{\frac{T}{K}}_{<1}\right]}_{>0} > 0\end{aligned}$$

This result confirms our graphical analysis. Since the derivative at T is greater than 0, this means that a positive perturbation of $S(t)$ at T will increase the stock even more, and therefore will push the system far from the original state. On the contrary, for K we have:

$$\begin{aligned} N'(K) &= r\left[1 - \frac{2K}{K} + \frac{T}{K}\right] \\ &= r\left[1 - 2 + \frac{T}{K}\right] \\ &= r\underbrace{\left[\underbrace{\frac{T}{K}}_{<1} - 1\right]}_{<0} < 0 \end{aligned}$$

Which again goes in the same direction as the graphical intuition ($K > T$ hence their ratio is less than 1). If we positively perturb the steady state at K, the stock of fish will decrease until we reach the previous state again.

Question: Do you know an easier way to check the sign of the derivative from the graph?

TD5

1 Review Questions

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c. An improvement in extractive technology always increases fish production if fishing is socially optimal.

- *Answer*

  True: From you notes (page 9), the expression for the total harvest in the social planner solution is $H_O^* = \frac{rK}{4}\left(1-\left(\frac{c}{p\alpha K}\right)^2\right)$. You can see that an increase in α will augment total harvest even without taking derivatives.

  Question: Don't you think the comparison between this question and the solution to point **b.** is interesting?

d. An improvement in extractive technology is always a bad thing from an environmental point of view.

- *Answer*

  True: We can check from the equilibrium expression of the stock of natural resources $S^* = \frac{c}{p\alpha}$. If α increase then the stock of natural resources decreases.

3 The Dynamics of a Fish Population with Threshold

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c. What is the natural growth of the fish population at t if $S(t) = 0$? Is it also an equilibrium?

To answer this question we just need to evaluate the growth rate in the point $S(t) = 0$.

$$N(0) = r(0-T)(1-0) = -rT$$

We should have expected this result, as we know that T is a threshold for the fish to grow and $S(t) = 0 < T$. Since the computed growth rate is negative and since the stock can not go lower than 0, we conclude that $S(t) = 0$ is also a steady state. Notice that $S(t)$ is the point in which the growth rate crosses the y axis, as shown in the picture below.

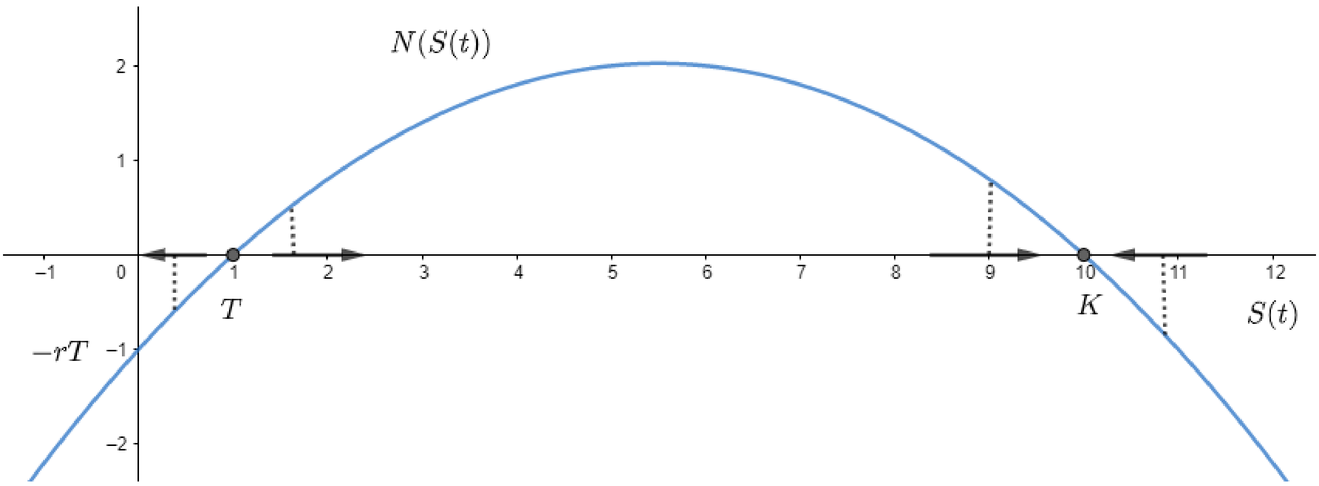

Figure 1: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$. Notice that $-rT = -1(1) = -1$, where the growth rate is negative and the stock is 0.

Question: Is this new steady state stable?

d. What is the maximum number of fish that can be caught per unit of time such that the fish population is constant? This is also called the maximum sustained yield. What is the fish stock $S(t)$ at this value?

To answer this question we must ask when the growth rate of fishes is the highest. This would allow us to capture the maximum number of fishes every time t and then obtain for $t + \varepsilon$ the greatest amount of growth so that we can always maximise our catches. Hence, we must maximise the growth rate of fishes with respect to the stock. We already have the derivative. To check for the maximum we need to find the value of $S(t)$ for which the derivative is equal to 0.

$$
\begin{aligned}
\frac{\partial N(S(t))}{\partial S(t)} = 0 \Leftrightarrow r\left[\left(1 - \frac{S(t)}{K}\right) + (S(t) - T)\left(-\frac{1}{K}\right)\right] = 0 \\
\Leftrightarrow \cancel{r}\left[1 - \frac{2S(t)}{K} + \frac{T}{K}\right] = 0 \\
\Leftrightarrow K - 2S(t) + T = 0 \\
\Leftrightarrow S(t) = \frac{K+T}{2}
\end{aligned}
$$

Now that we have the stock of fishes that maximises growth, we can ask by how much fish grows for this value of the stock. Of course, to answer this question we just need to plug the value we just found in the growth rate.

$$
\begin{aligned}
N\left(\frac{K+T}{2}\right) &= r\left(\frac{K+T}{2} - T\right)\left(1 - \frac{K+T}{2K}\right) \\
&= r\left(\frac{K+T-2T}{2}\right)\left(\frac{2K-K-T}{2K}\right) \\
&= r\left(\frac{K-T}{2}\right)\left(\frac{K-T}{2K}\right) \\
&= r\frac{(K-T)^2}{4K}
\end{aligned}
$$

This expression tells us by how much fish we can catch for growth to always be at its maximum.

e. Graph the fish growth function $S(t)$. Place all the elements previously computed on the graph.

We already did a big part of the graph, the one below has also the answers to the last question.

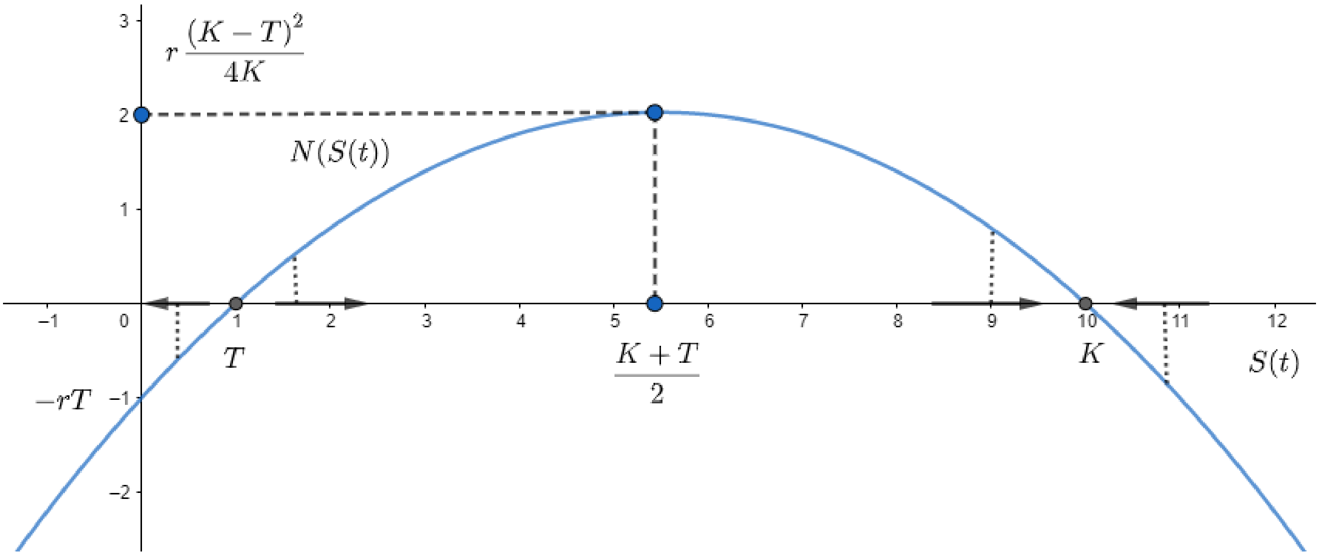

Figure 2: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$. Here $S^*(t) = \frac{10+1}{2} = 5.5$ and $N(S^*(t)) = \frac{9^2}{40} \approx 2$.

f. If there are B boats catching fishes, their total catches are $H(t) = \alpha BS(t)$. The net growth rate (the law of motion) of the stock is $\dot{S}(t) = N(t) - H(t)$. With B boats in the ocean, what is (are) the steady-state population(s) of fish?

First, notice that $\dot{S}_t$ is just notation for $\frac{\partial S(t)}{\partial t}$, the derivative of the stock of fish with respect to time. It is the equivalent of the law of motion of the Solow - Swan growth model, so you should treat it exactly as we did with that model. This observation helps us answering this question. In fact, the steady state population of fish is characterised by setting its growth rate equal to 0, which is the same as saying that $N(t) = H(t) - N(t) = 0 \implies H(t) = (t)$. We are looking for solutions of a quadratic equation, hence I rearrange terms to employ the classical formula.

$$
\begin{aligned}
\dot{S}_t = 0 &\Leftrightarrow r(S(t)-T)\left(1-\frac{S(t)}{K}\right) - \alpha BS(t) = 0 \\
&\Leftrightarrow rS(t) - rT - \frac{rS(t)^2}{K} - \frac{rTS(t)}{K} - \alpha BS(t) = 0 \\
&\Leftrightarrow -\frac{rS(t)^2}{K} + S(t)\left(r + \frac{rT}{K} - \alpha B\right) - rT = 0 \\
&\Leftrightarrow S(t)^2\frac{r}{K} - S(t)\left(r + \frac{rT}{K} - \alpha B\right) + rT = 0 \\
&\Leftrightarrow S(t)^2 - S(t)\left(K + T - \frac{\alpha BK}{r}\right) + TK = 0 \text{ Multiply everything by } \frac{K}{r} \\
&\Leftrightarrow S(t)^2 + S(t)\left(\frac{\alpha BK}{r} - K - T\right) + TK = 0
\end{aligned}
$$

This is a second order degree equation of which we have to find the roots by the usual formula. There are two solutions that we label S_U and S_S (you will soon see why).

$$
S_U = \frac{K + T - \frac{\alpha BK}{r} - \sqrt{\left(\frac{\alpha BK}{r} - K-\right)^2 - 4TK}}{2}
$$

$$
S_S = \frac{K + T - \frac{\alpha BK}{r} + \sqrt{\left(\frac{\alpha BK}{r} - K - T\right)^2 - 4TK}}{2}
$$

These are the two steady state populations of fish.

g. Graph the dynamics of the stock with resource extraction and identify the equilibrium population(s) of fish. Show with arrows how population dynamics pushes S to increase or decrease.

Here the picture where I added the solutions computed in the previous point. Notice that here the growth rate is given by the difference $N(t) - H(t)$. Hence, to check the stability of steady states you have to see which one is above the other. Consider S_S in the picture, as an example. If you perturb it towards the right (i.e. $S_S + \varepsilon$), you see that $H(t)$ is greater than $N(t)$, hence $\dot{S}(t)$ is negative and we get back to S_S. By performing the same reasoning for any perturbation you can easily see that S_U is unstable while S_S is stable.

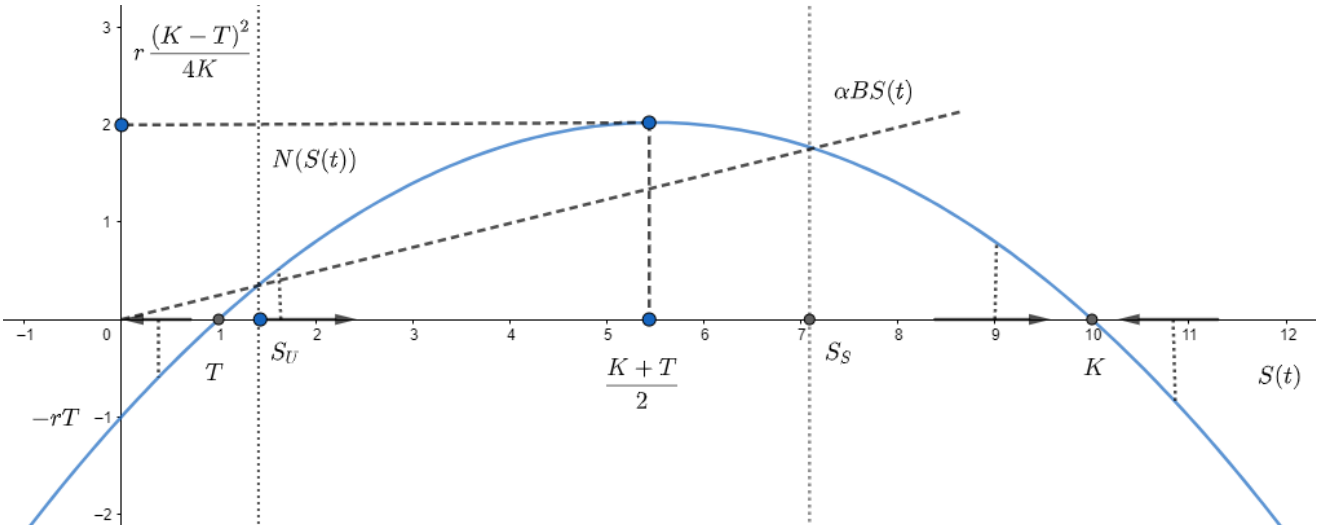

Figure 3: Same graph as before with S_U and S_S. Here I picked $\alpha = \frac{1}{8}$ and $B = 4$.

h. Is there an intensity of fishing (αB) so high that no sustainable fishing is possible? What is it?

To answer this question it is enough to notice that if you increase αB by a lot then the line $\alpha BS(t)$ will not cross $N(S(t))$ anymore, which means that no sustainable fishing is possible. This will happen when there is no solution to the previous second degree equation, that is when the quantity below the square root is negative (imaginary solution). Therefore we just have to check when this condition is satisfied. We search again for the solutions of a quadratic equation in αB.

$$\left(\frac{\alpha BK}{r} - K - T\right)^2 - 4TK < 0$$

$$\pm\left(\frac{\alpha BK}{r} - K - T\right) < \sqrt{4TK}$$

$$\frac{\alpha BK}{r} > T + K - 2\sqrt{KT} \quad \text{In the minus (-) case}$$

$$\alpha B > \frac{r}{K}\left(T + K - 2\sqrt{KT}\right)$$

$$\alpha B > \frac{r}{K}\left(\sqrt{T} - \sqrt{K}\right)^2$$

$$\alpha B > r\left(\frac{\sqrt{K}}{\sqrt{K}} - \frac{\sqrt{T}}{\sqrt{K}}\right)^2$$

$$\alpha B > r\left(1 - \sqrt{\frac{T}{K}}\right)^2$$

If you perform the same operation in the plus (+) case you will find that $\alpha B < r\left(1 + \sqrt{\frac{T}{K}}\right)^2$, hence we obtain that no sustainable fishing is possible when $r\left(1 - \sqrt{\frac{T}{K}}\right)^2 < \alpha B < r\left(1 + \sqrt{\frac{T}{K}}\right)^2$. Just as a remark, notice that in the numerical example from which I plotted the graph this condition is satisfied and therefore we have the two solutions.

Question: Can you guess what happens if $\alpha B > r\left(1+\sqrt{\frac{T}{K}}\right)^2$?

Question: Can you think about other methods to do this point?

i. The profit from a boat is $\pi(t) = p\alpha S(t) - c$. If there is free entry, fishing boats will enter as long as profits are positive. What is the free market equilibrium value of the stock S_F^* in the steady state.

If boats will continue to enter as long as profits are positive, then they will stop when profits are 0. Therefore, as in class, to find the free market equilibrium value of $S(t)$ we just need to check when this condition is satisfied.

$$\pi(t) = 0 \Leftrightarrow p\alpha S_F^* - c = 0 \Leftrightarrow S_F^* = \frac{c}{p\alpha}$$

However, notice that in class we had $T = 0$, and since $\frac{c}{p\alpha}$ is always weakly greater than 0 we never had any problem. In this case, if $\frac{c}{p\alpha} < T$ the growth is negative and the stock goes to 0.

TD6

4 Taxation to Obtain Optimum Resource Extraction

This exercise make you compute Pigouvian taxes. These kind of taxes are classical in the economics literature. Their aim is to correct for externalities that affect the market outcome without passing trough the channel of prices. In the fishing model an increase of boat affects the growth of natural resources in a way that is not transmitted to the market with the price p. Recall that the free market optimal number of boats is $B_F^* = \frac{r}{\alpha}\left(1 - \frac{c}{p\alpha K}\right)$, while from a social planner perspective we should have $B_O^* = \frac{B_F^*}{2}$.

a. Show that the optimal number of boats could be obtained by a tax per boat $t = \frac{p\alpha K - c}{2}$.

There are two ways to answer this question. The first way, which is the one you will have in the professor's solution, is to ask "which is the value of t such that if boats pay the new cost $c' = c + t$ then $B_O^* = B_{F'}^*$?". This means solving the following equation:

$$B_{F'}^* = \frac{r}{\alpha}\left(1 - \frac{c+t}{p\alpha K}\right) = \frac{r}{2\alpha}\left(1 - \frac{c}{p\alpha K}\right) = B_O^*$$

and realise that $t = \frac{p\alpha K - c}{2}$. Since you have this way already explained in your solution I will show you the second way. I think it is less smart but more algorithmic, in case you do not have the intuition to frame the problem in the way I just exposed.

The second way amounts to perform the same step you did in the class, but the profits are $\pi(t) = \alpha pS(t) - (c+t) = \alpha pS(t) - (c + \frac{p\alpha K - c}{2})$ and realise that the free market equilibrium boats are equal to the social optimum. In optimum we must always have that profits are equal to 0, therefore:

$$\alpha p S^*(B^*) - \left(c + \overbrace{\frac{p\alpha K - c}{2}}^{t} \right) = 0$$

$$\alpha p K \left(1 - \frac{\alpha B}{r}\right) - \left(c + \frac{p\alpha K - c}{2}\right) = 0$$

$$p\alpha K - \frac{p\alpha K \alpha B}{r} = c + \frac{p\alpha K - c}{2}$$

$$1 - \frac{c}{p\alpha K} - \frac{1}{2} + \frac{c}{2p\alpha K} = \frac{\alpha B}{r}$$

$$\frac{1}{2} - \frac{c}{2p\alpha K} = \frac{\alpha B}{r}$$

$$\frac{1}{2}\left(1 - \frac{c}{p\alpha K}\right)\frac{r}{\alpha} = B^*_{F'} = B^*_O$$

Which is the result we wanted.

b. Illustrate this tax on the graph of the revenue of the fishing industry.

The change in marginal revenues due to the introduction of the tax affects the point in which this line intersect the marginal cost c. You could also interpret it by saying that the new marginal cost is $c + t$ and the optimality condition requires the blue line to intersect with the new marginal cost.

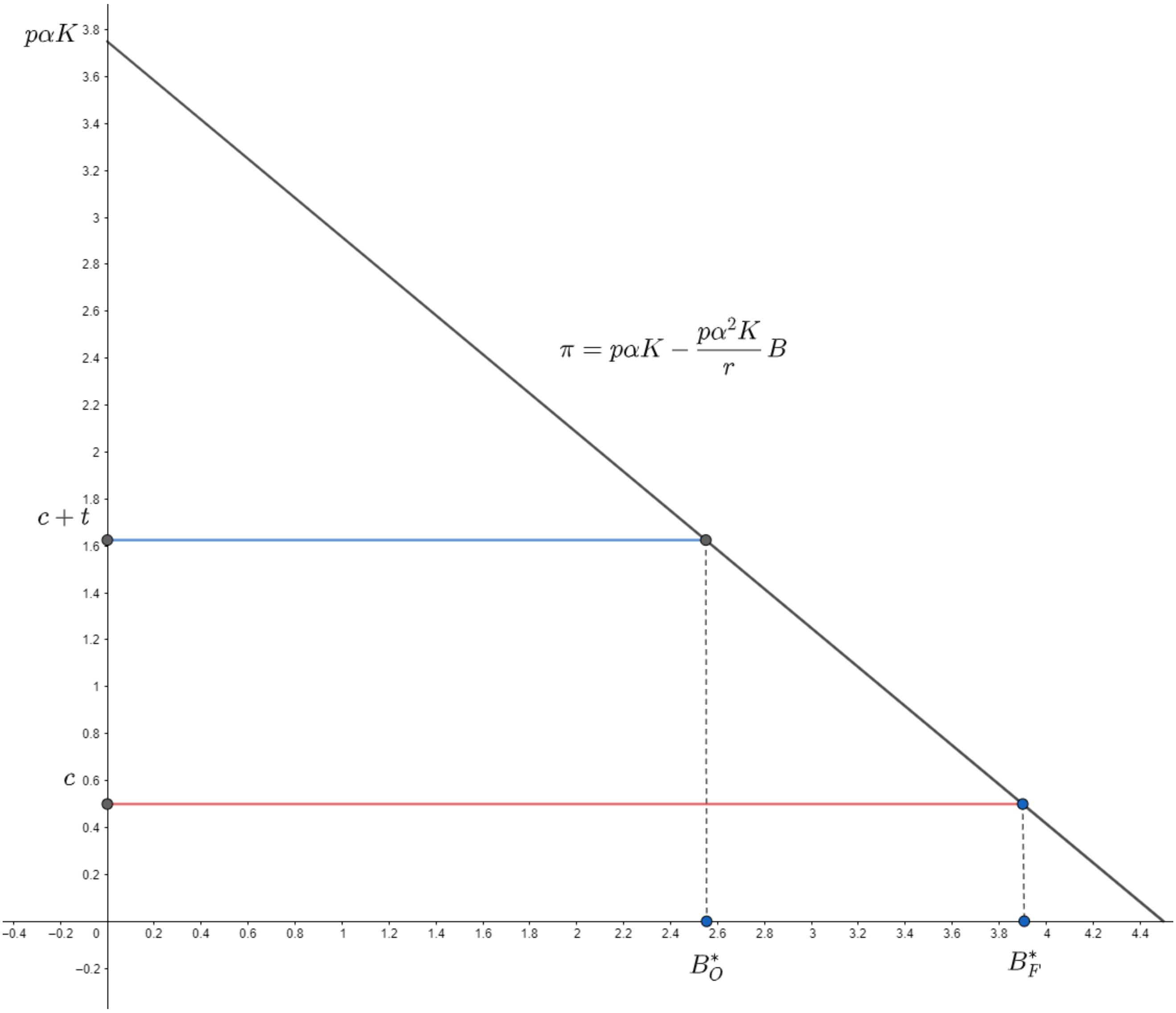

Figure 1: Profits and marginal costs for $p = 3$, $c = 0.5$ and, as before, $\alpha = \frac{1}{8}$, $K = 10$, $r = 1$.

c. Show that an ad valorem tax on fish sales of $\tau = \frac{p\alpha K - c}{p\alpha K + c}$ would achieve the optimum as well.

Exactly as before, we can solve this problem in two ways. The first one is to compute the τ proportional tax on π that would make $B_{F'}^* = B_0^*$. This is equivalent to solving the following equation:

$$B_{F'}^* = \frac{r}{\alpha}\left(1 - \frac{c}{p(1-\tau)\alpha K}\right) = \frac{r}{2\alpha}\left(1 - \frac{c}{p\alpha K}\right) = B_O^*$$

You have the detail of this method in the professor's solution.

The second method follows the same reasoning of the previous point. We just change the expression for profits and find the optimal number of boats in free markets $B_{F'}^*$. The new profits are $\pi(t) = p(1-\tau)\alpha S(t) - c = p\left(1 - \frac{p\alpha K - c}{p\alpha K + c}\right)\alpha S(t) - c$ and we must set them equal to 0.

$$p\left(1-\overbrace{\frac{p\alpha K-c}{p\alpha K+c}}^{\tau}\right)\alpha S(B^*)-c=0$$

$$p\left(1-\frac{p\alpha K-c}{p\alpha K+c}\right)\alpha K\left(1-\frac{\alpha B}{r}\right)-c=0$$

$$p\left(\frac{2c}{p\alpha K+c}\right)\alpha K\left(1-\frac{\alpha B}{r}\right)-c=0$$

$$\frac{p\alpha K2c}{p\alpha K+c}-\frac{p\alpha K2c\alpha B}{r(p\alpha K+c)}-c=0$$

$$\frac{p\alpha K2c}{p\alpha K+c}-c=\frac{p\alpha K2c\alpha B}{r(p\alpha K+c)}$$

$$1-\frac{\cancel{c}(p\alpha K+c)}{p\alpha K2\cancel{c}}=\frac{\alpha B}{r}$$

$$1-\frac{1}{2}-\frac{c}{p\alpha K2}=\frac{\alpha B}{r}$$

$$\frac{1}{2}\left(1-\frac{c}{p\alpha K}\right)\frac{r}{\alpha}=B^*_{F'}=B^*_O$$

Which is again the solution we wanted.

Question: How does the graph looks like here?

5 Equilibrium on Easter Island

In this exercise, we just have to reason with the phase diagram to get the answers. First, let's derive again the fundamental equations. Here we work in a system in which there is both growth of a natural resource, as in the fishing model, and population growth. As we just saw in the review question, the net growth of the natural resource is given by its natural growth rate minus the harvest (exactly as in the fishing model).

$$\begin{aligned}\dot{S}(t)&=\underbrace{rS(t)\left(1-\frac{S(t)}{K}\right)}_{\text{Natural growth}}-\underbrace{\alpha\beta S(t)L(t)}_{\text{Harvest}}\\&=\left(r\left(1-\frac{S(t)}{K}\right)-\alpha\beta L(t)\right)S(t)\end{aligned}$$

As for population growth, we can easily recover it by using the fundamental parameters of the model. We have that people die at a rate d and born at a rate b plus a percentage of per capita harvest $\phi\frac{H(t)}{L(t)}$. Of course, do not forget that the growth itself at time t depends on how many people $L(t)$ we have.

$$\begin{aligned}\dot{L}(t)&=\left(b-d+\phi\frac{H(t)}{L(t)}\right)L(t)\\&=\left(b-d+\phi\frac{\alpha\beta S(t)\cancel{L(t)}}{\cancel{L(t)}}\right)L(t)\\&=(b-d+\phi\alpha\beta S(t))L(t)\end{aligned}$$

These are the two **laws of motion** of the variables of interest in our model, $S(t)$ and $L(t)$. In the previous model with fish extraction we were used to graph the growth rate and see the points which constituted the stable states. Here the problem is a bit harder, as we have two variables with two laws of motion, not only one. Graphing the two growth rates separately will not help much in understanding the dynamics of the system. Hence, instead of employing a graph which on one axis has $S(t)$ and on the other one has its growth rate, we use a graph with the two variables of interest, both $S(t)$ and $L(t)$. But what do we represent?

We have a system with two variables and two growth rates, which means that each law of motion will have more than one couple of $S(t)$ and $L(t)$ in which it is zero. Take $\dot{S}(t)$, as an example. In the previous exercise we found the value of $S(t)$ (the fish stock) for which its growth was equal to 0. We do the same here, but we will not find a value of $S(t)$, rather a relation between $S(t)$ and $L(t)$! Let's do it, when is $\dot{S}(t)$ equal to 0?

$$\dot{S}(t) = 0 \Leftrightarrow \left(r\left(1 - \frac{S(t)}{K}\right) - \alpha\beta L(t)\right)S(t) = 0$$

This equation has two solutions: the first is $S(t) = 0$, for any value of $L(t)$, while the second is $r\left(1 - \frac{S(t)}{K}\right) - \alpha\beta L(t) = 0$, which gives us $S(t) = K - \frac{\alpha\beta K}{r}L(t)$. You see now that $\dot{S}(t)$ is not zero only for some values of $S(t)$, but for couples of values of $S(t)$ and $L(t)$. In all the points of the graph with axis $(L(t), S(t))$ in which $S(t) = K - \frac{\alpha\beta K}{r}L(t)$, the law of motion of S is equal to 0.

We must now repeat the same exercise with $\dot{L}(t)$.

$$\dot{L}(t) = 0 \Leftrightarrow (b - d + \phi\alpha\beta S(t))L(t)$$

We have two solutions here too: the law of motion is null when $L(t) = 0$, for any value of $S(t)$, but also when $b - d + \phi\alpha\beta S(t) = 0$ which gives $S(t) = \frac{d-b}{\phi\alpha\beta}$.

Hence, the equations that give us the set of points $S(t)$ and $L(t)$ in which the two law of motions are zero are given by:

$$S^* = K - \frac{K\alpha\beta}{r}L^* \qquad and \qquad S^* = 0$$

$$S^* = \frac{d-b}{\phi\alpha\beta} \qquad and \qquad L^* = 0$$

When our variables respect these conditions there is no movement of S (first equation) or L (second equation). The system completely stops when we are in the steady state, which in this case is when both these equations are satisfied, i.e. neither S nor L are moving. Every time we are not in a point in which these two lines intersect we have movement in the system. If we forget about the 0 solutions we can graph the two lines with the dynamics when we are out of the steady state here.

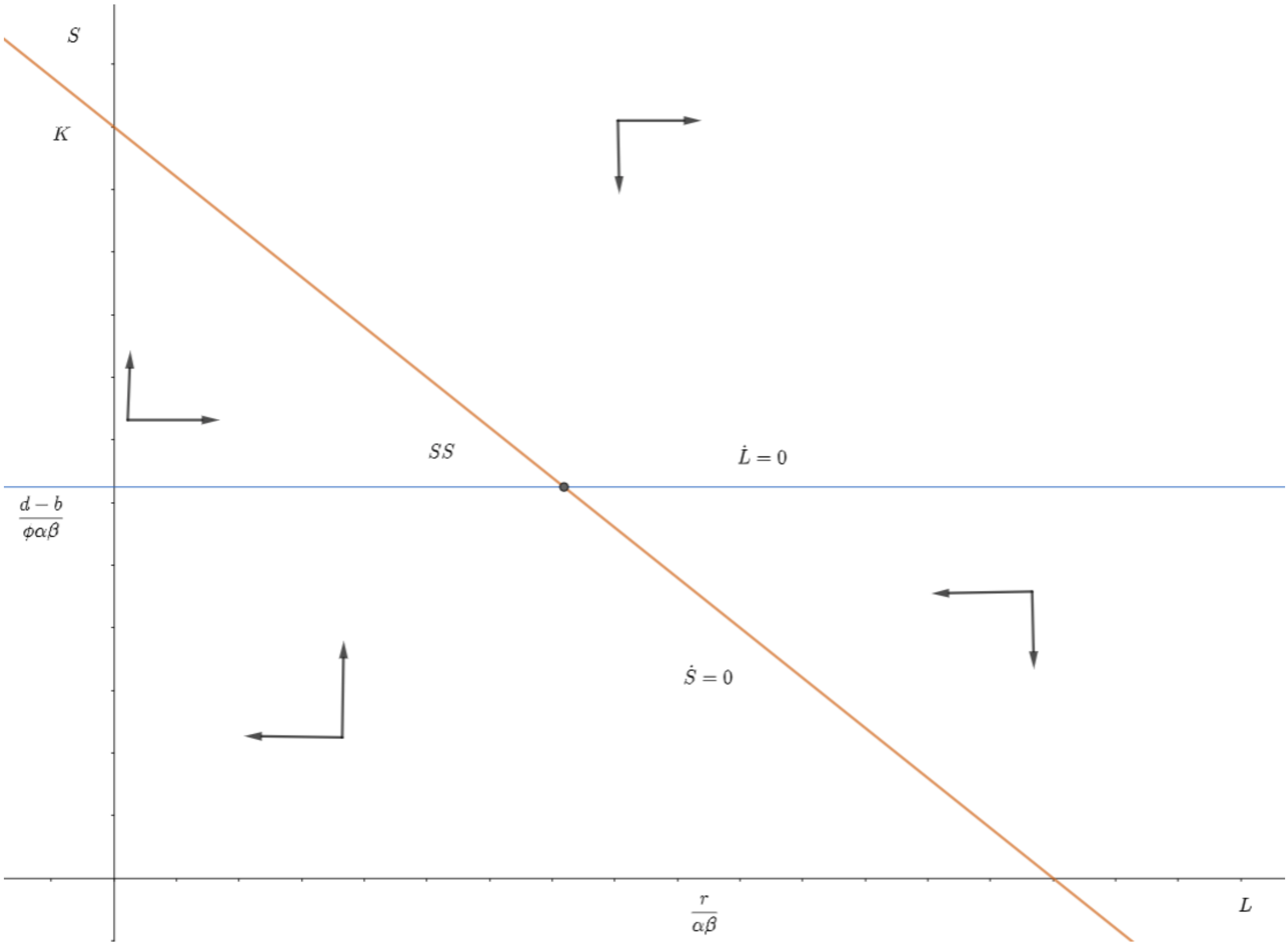

Figure 2: Phase diagram with $r = 0.04$, $\alpha = 10^{-6}$, $b - d = 0.1$, $\phi = 4$, $\beta = 0.4$, $K = 12000$.

TD7

1 Review Questions

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e. In the model of the Easter Island seen in class, there is a threshold amount of resource such that population is growing above that threshold, and decreasing below that threshold.

- *Answer*

  True: You can easily see it from the phase diagram. We have a line along which S is fixed, above that line it increases and below it decreases. The expression of that line is $S = \frac{d-b}{\phi\alpha\beta}$, therefore if S is greater than $\frac{d-b}{\phi\alpha\beta}$ the population is growing, otherwise it is decreasing.

f. In the model of the Easter Island seen in class, the net growth of the resource (i.e. net of harvesting by humans) is decreasing in the numbers of humans present in the ecosystem, for a given value of the amount of the resource.

- *Answer*

  True: The net growth is given by the replenish of the resource minus the harvest, which is the part affected by the number of humans (population).

  $$\dot{S}(t) = \underbrace{rS(t)\left(1 - \frac{S(t)}{K}\right)}_{\text{Natural growth}} - \underbrace{\alpha\beta S(t)L(t)}_{\text{Harvest}}$$

  We can clearly see that an increase in $L(t)$ decreases the net growth. If you want to be more precise its enough to take the derivative with respect to $L(t)$:

  $$\frac{\partial \dot{S}(t)}{\partial L(t)} = -\alpha\beta S(t) < 0$$

  What you get is how much growth is smaller after an infinitesimal change in $L(t)$.

5 Equilibrium on Easter Island

In this exercise, we just have to reason with the phase diagram to get the answers. In the rest of the solution, I indicate with dashed lines the old equations, while continuous lines represents the new ones after the variable changes. We assume we are in the nontrivial internal steady state.

a. What happens if r goes up? Show on the graph the change to each conditions and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.

Let's start by breaking down all the components of the relevant equations.

$$S^* = \underbrace{K}_{intercept} - \underbrace{\frac{K\alpha\beta}{r}}_{slope} L^*$$

$$S^* = \underbrace{\frac{d-b}{\phi\alpha\beta}}_{intercept}$$

We can see that r only appears in one of the two equations, namely $S^* = K - \frac{K\alpha\beta}{r}L^*$. Since r is positively affecting its slope, but not the intercept, we have to rotate it counter-clockwise (as r increases). The other equation is not affected.

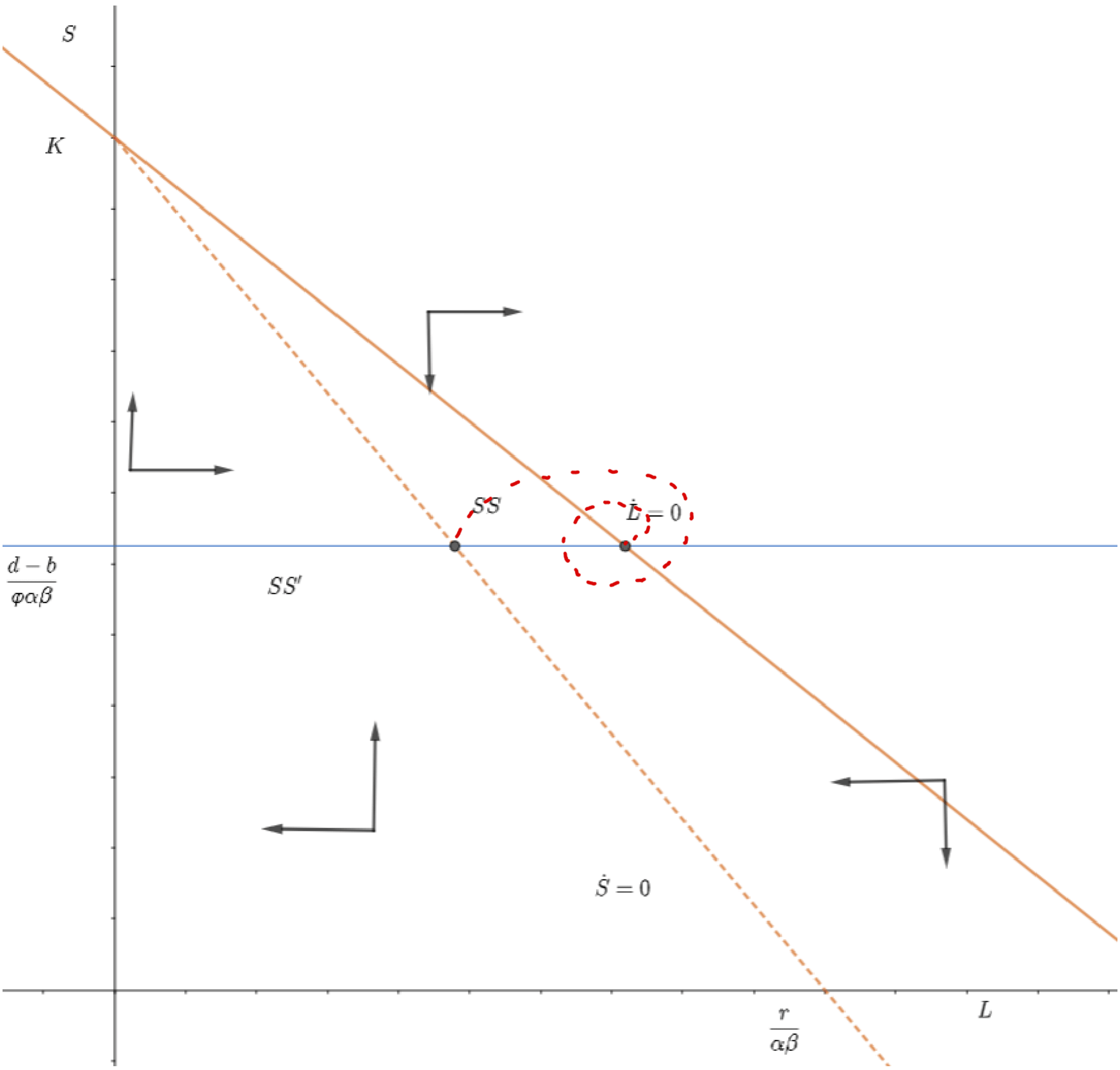


Figure 1: Phase diagram with new $r' = 0.06$.

What we see is that in the new steady state S^* is not affected, but L^* is higher. This is due to the fact that the increase of availability of resources is completely offset by the increase in population, which was possible because of the increase of the regeneration rate.

b. What happens if α goes up? Show on the graph the change to each conditions and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.

The rate α appears again in the intercept of the locus for $\dot{S} = 0$. However, in this case the rotation is clockwise as it is negatively affected by an increase in α. Moreover, α is also in the intercept for $\dot{L} = 0$, which goes down due to their negative relationship.

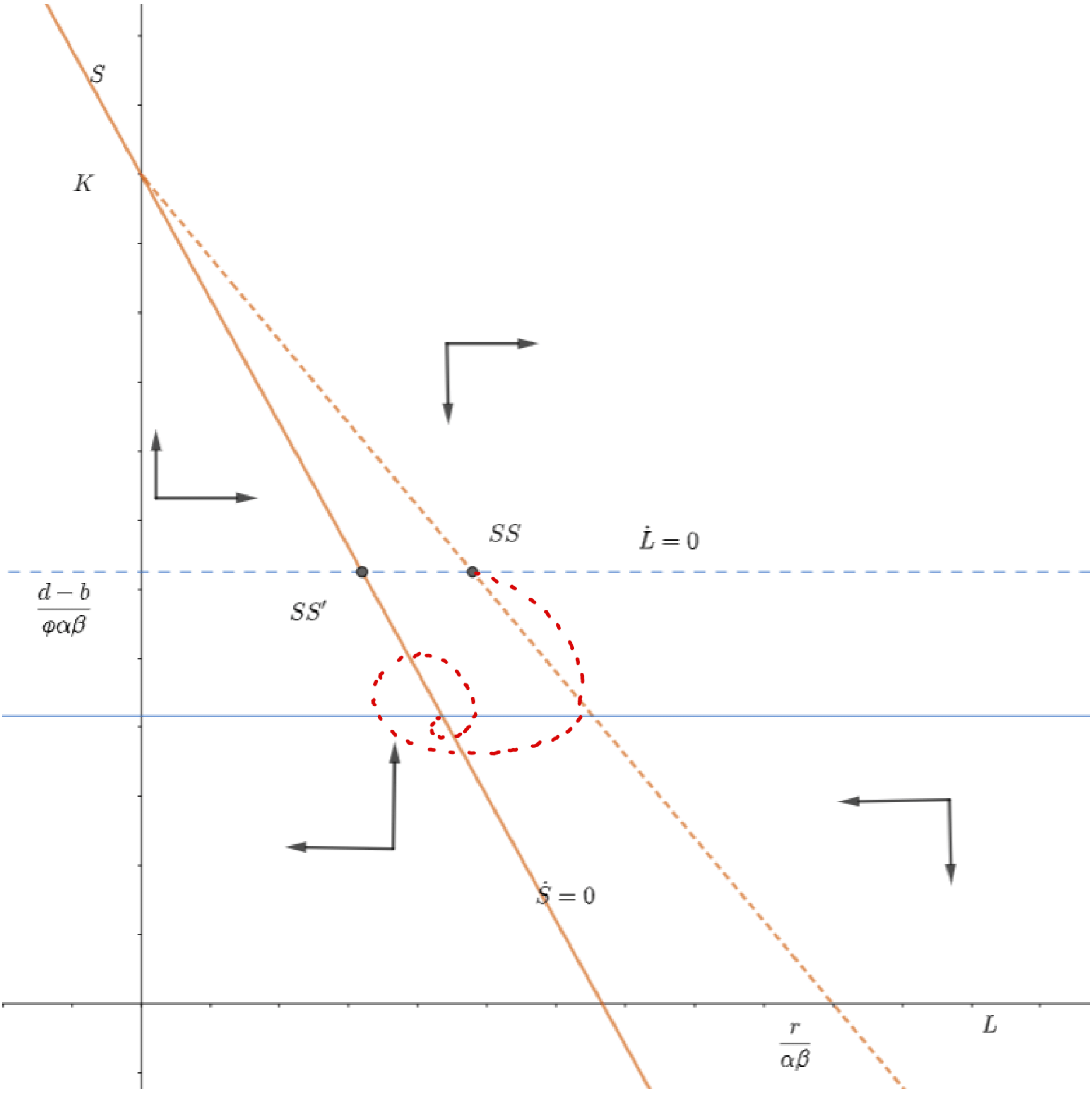


Figure 2: Phase diagram with new α' that leads to a decrease in L.

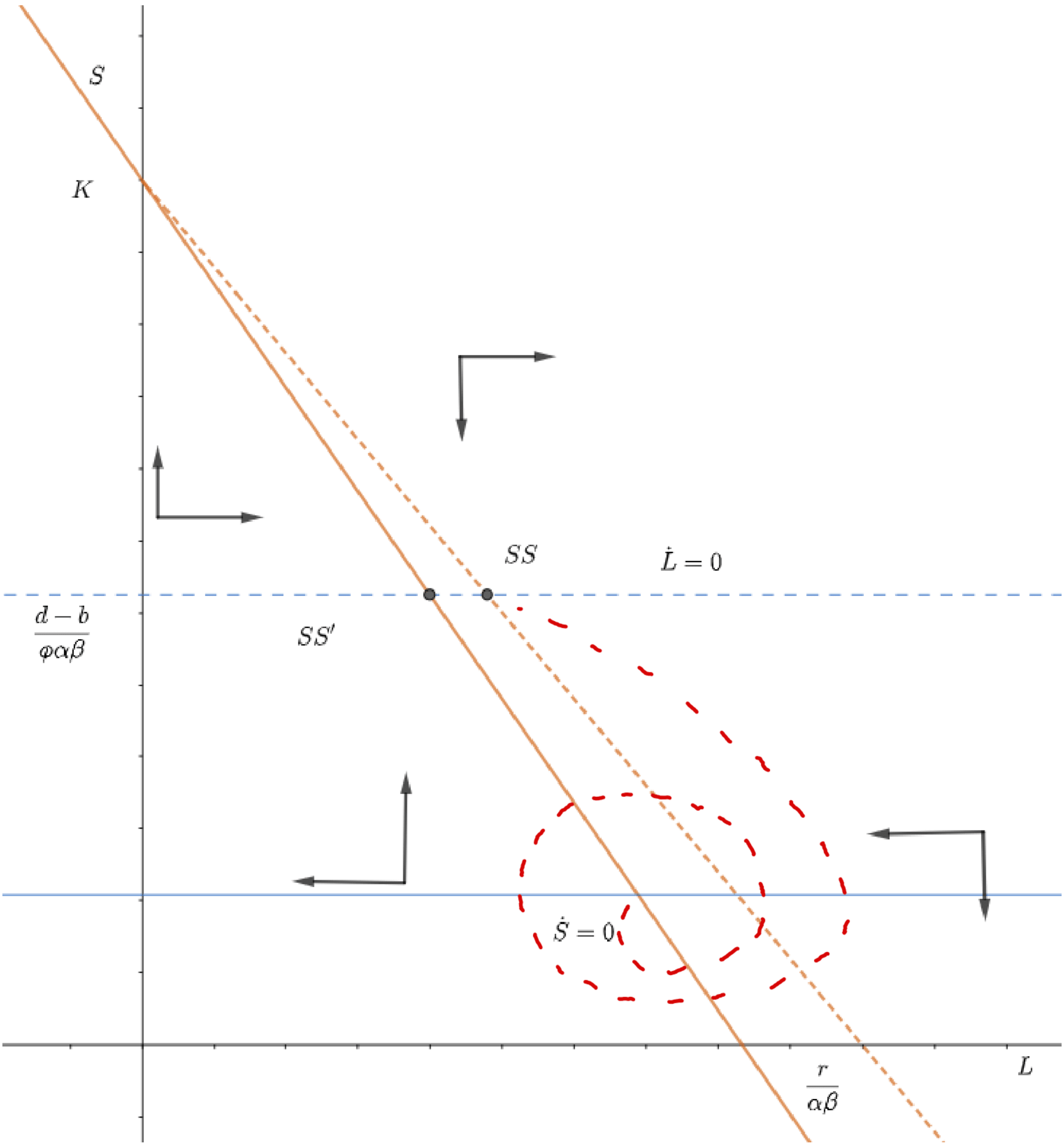


Figure 3: Phase diagram with new α' that leads to an increase in L.

In the new steady state we have unambiguously lower S^*. The impact on L^* depends on the entity of the increase of α, as illustrated by the pictures above. More efficient farming means that we need less workers to keep the amount of resources fixed (clockwise rotation). On the other hand, given the increase efficiency we need less resources to keep population constant (horizontal line decrease).

c. What happens if K goes down? Show on the graph the change to each conditions and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.

The capacity K only appears in the intercept and slope of $\dot{S} = 0$, which then must rotate counter clockwise.

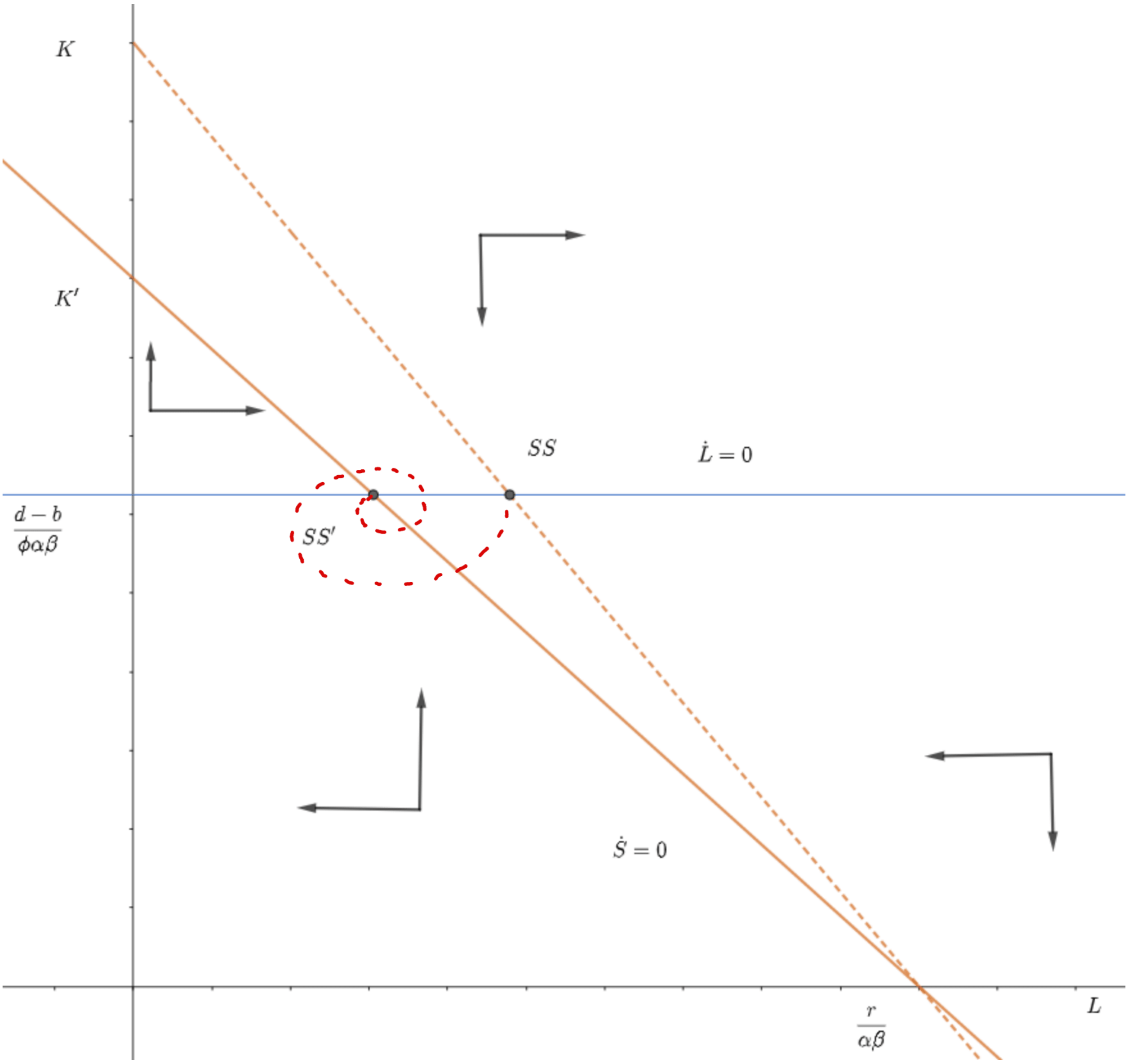


Figure 4: Phase diagram with new $K' = 9000$.

This time S^* is not affected while the impact on L^* is unambiguous. Due to the decrease in maximum capacity, the stock of resources grows slower. You can see this from the growth rate $\dot{S}$. This implies that L^* is lower than before.

TD8

1 Review

---

a. Before the demographic transition, increases in income per capita always caused an increase in the growth rate of population.

- *Answer*

  True: As you found in your lecture, in the pre industrialised world technological progress and land expansion caused temporary increase in income per capita which in turn increased the growth rate of population. As an example see the graph at page 3.

b. In the contemporary world, an increase in income per capita is associated to a decrease in the growth rate of population.

- *Answer*

  True: The graph at page 12 of your lecture notes is self explanatory. Higher average income correlates with lower births per woman. An explanation taken from your lecture notes could be the effects of urbanization.

c. Decreases in the various measures of fertility came after decreases in mortality.

- *Answer*

  True: You can see from the graph at page 11 of your lecture notes that the fertility rate has decreased from 1950. For sure the mortality rate had significantly declined when compared to pre 1900 times. This statement is also consistent with the fact that fertility rates are lower in developed countries where the mortality rate is lower.

d. The demographic transition is now over for most of the world population.

- *Answer*

  True: Quoting from page 11 of your notes "*as of 2017, more than 80% of the world fertility was already at or below replacement rate*".

e. In the model of the Malthusian regime seen in class, an exogenous increase in the birth rate translates into a lower level of steady-state income per capita.

- *Answer*

  Maybe: If this shock implies that $b(y_t)$ is higher, keeping other things fixed, then the statement is true, you can see it from the graph presented here. Clearly, if the shock is such that $b\left(y_t\right)$ is lower, then income per capita will be higher.

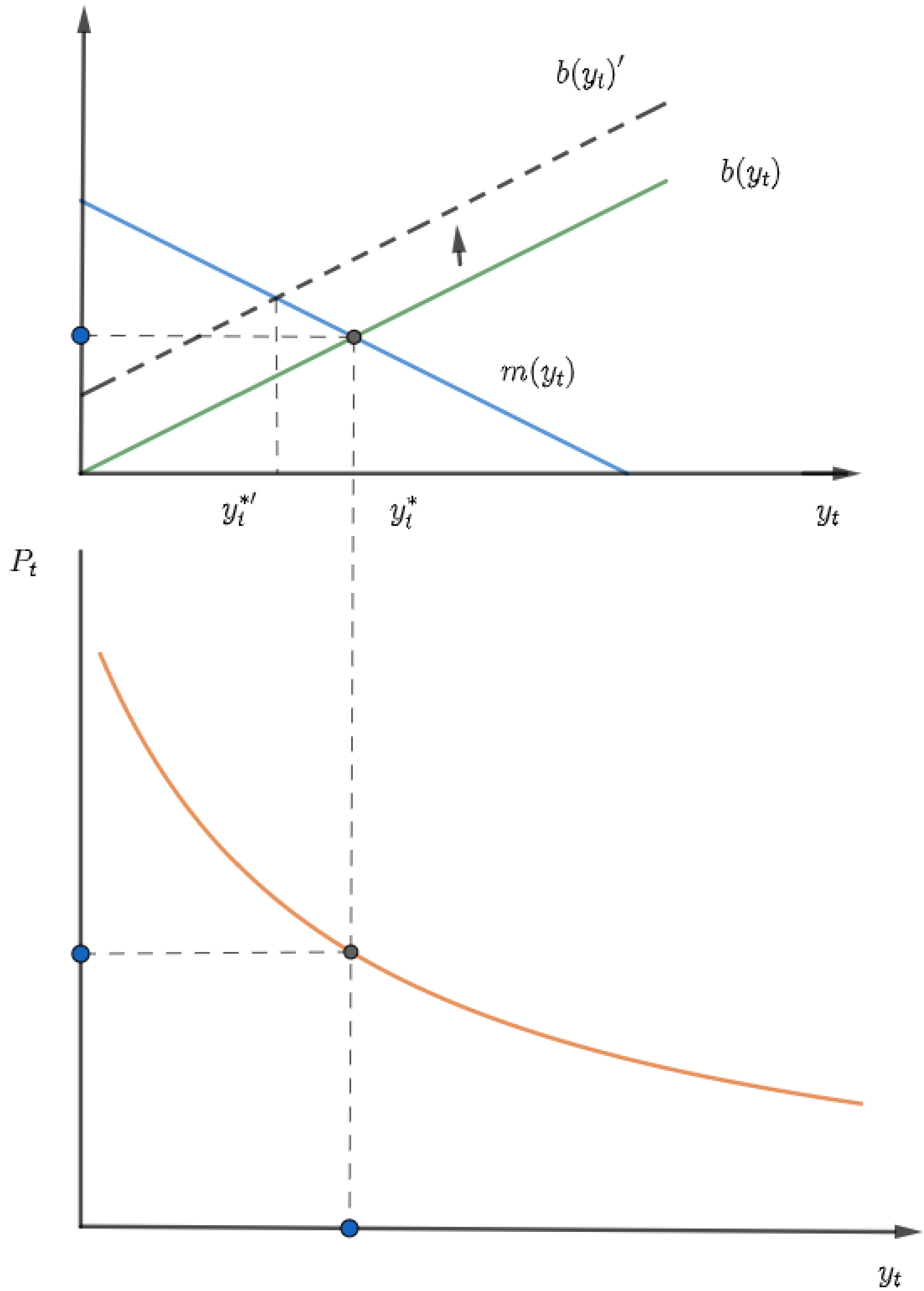


Figure 1: Shock to $b\,(y_t)$.

2 The Malthusian Regime

This exercise studies a particular example of the model seen in class, with specifications for the primitives of the model. I report them here.

The birth rate is given by:

$$b(y_t) = \alpha_b + \beta_b y_t \tag{1}$$

The mortality rate is:

$$m(y_t) = \alpha_m - \beta_m y_t \tag{2}$$

The production function is:

$$Y(P_t) = \alpha_y + \beta_y P_t \tag{3}$$

Y is the total production (or income), P is the population, y is income per capita. It is assumed that all the population works and that there is no immigration nor emigration. All coefficients of the model are positive. It is further assumed that $\alpha_m > \alpha_b$ and $\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m) > 0$, you will soon understand why.

a. Discuss equations (1) and (3): how do they relate to the model seen in the lecture?

In your lectures you saw that the birth rate depends positively on income per capita, which translates into $b(y_t)$ with $\frac{\partial b(y_t)}{\partial y_t} > 0$. This condition is indeed respected in the specific function we have in this exercise, as $\frac{\partial b(y_t)}{\partial y_t} = \beta_b > 0$.

As for the technology, the condition you had in the lecture were positive marginal product $\frac{\partial Y(P_t)}{\partial P_t} > 0$ and decreasing returns $\frac{\partial^2 Y(P_t)}{\partial P_t^2} < 0$. It is easy to check that the first holds while the second does not. In fact, $\frac{\partial Y(P_t)}{\partial P_t} = \beta_y > 0$ and $\frac{\partial^2 Y(P_t)}{\partial P_t^2} = 0 \not< 0$. This production function is similar to what you saw in TD2 in the Solow Model, it is affine! Since it is a line it does not have decreasing returns. Will this be a problem for the existence of a steady state? Let's see…

b. Compute the marginal and average productivity of labor. Comment.

To compute average and marginal productivity, we first need productivity. As usual we divide by the population P_t.

$$\begin{aligned} y(P_t) &= \frac{Y(P_t)}{P_t} \\ &= \frac{\alpha_y + \beta_y P_t}{P_t} \\ &= \frac{\alpha_y}{P_t} + \beta_y \end{aligned}$$

Marginal productivity indicates how much more productive we are by increasing P_t by an infinitesimal amount. By taking the derivative we get:

$$\frac{\partial y(P_t)}{\partial P_t} = -\frac{\alpha_y}{P_t^2} < 0$$

Hence, by adding labour we become less productive. As for average productivity we just have to divide by the number of workers

$$\frac{y(P_t)}{P_t} = \frac{\alpha_y}{P_t^2} + \frac{\beta_y}{P_t}$$

Nothing special here, as we increase the number of workers the productivity per worker decreases.

c. Compute the steady-state level of total income, per capita income and population. Show graphically how those steady-state values are determined.

The law of motion of population in this model is $\dot{P} = [(b(y_t) - m(y_t)]P$. As usual, we are in steady state when the law of motion is equal to 0, i.e. population is not growing $\dot{P} = 0$. This when the mortality rate is equal to the birth rate (actually, also when $P = 0$, but we are not interested in this case). We have to set $b(y_t) = m(y_t)$ to get:

$$\begin{aligned}
\alpha_b + \beta_b y^* &= \alpha_m - \beta_m y^* \\
y^*(\beta_b + \beta_m) &= \alpha_m - \alpha_b \\
y^* &= \frac{\alpha_m - \alpha_b}{\beta_b + \beta_m}
\end{aligned}$$

which is the steady state level of per capita income. You should see now why we assumed $\alpha_m > \alpha_b$. We still need to compute the steady-state level of total income and population. Total income is itself a function of the population, so we have to find the steady state of this one first. We can rely on y^* which we just found.

$$\begin{aligned}
y^* &= \frac{\alpha_y}{P^*} + \beta_y \\
\frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} &= \frac{\alpha_y}{P^*} + \beta_y \\
\frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} - \beta_y &= \frac{\alpha_y}{P^*} \\
P^*\left(\frac{\alpha_m - \alpha_b}{\beta_m + \beta_b} - \beta_y\right) &= \alpha_y \\
P^*\left(\frac{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}{\beta_m + \beta_b}\right) &= \alpha_y \\
P^* &= \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}
\end{aligned}$$

We obtain a positive P^* since we assumed $\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m) > 0$. Now that we have y^* and P^* we are ready to compute the steady state level of total income Y^*. Its expression is given in the text:

$$
\begin{aligned}
Y^* = Y(P^*) &= \alpha_y + \beta_y P^* \\
&= \alpha_y + \beta_y \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} \\
&= \frac{\alpha_y\alpha_m - \alpha_y\alpha_b - \cancel{\alpha_y\beta_y\beta_b} - \cancel{\alpha_y\beta_y\beta_m} + \cancel{\alpha_y\beta_y\beta_b} + \cancel{\alpha_y\beta_y\beta_m}}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} \\
Y^* &= \frac{\alpha_y(\alpha_m - \alpha_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}
\end{aligned}
$$

In the following figure I represented the equilibrium for specific values of the parameters.

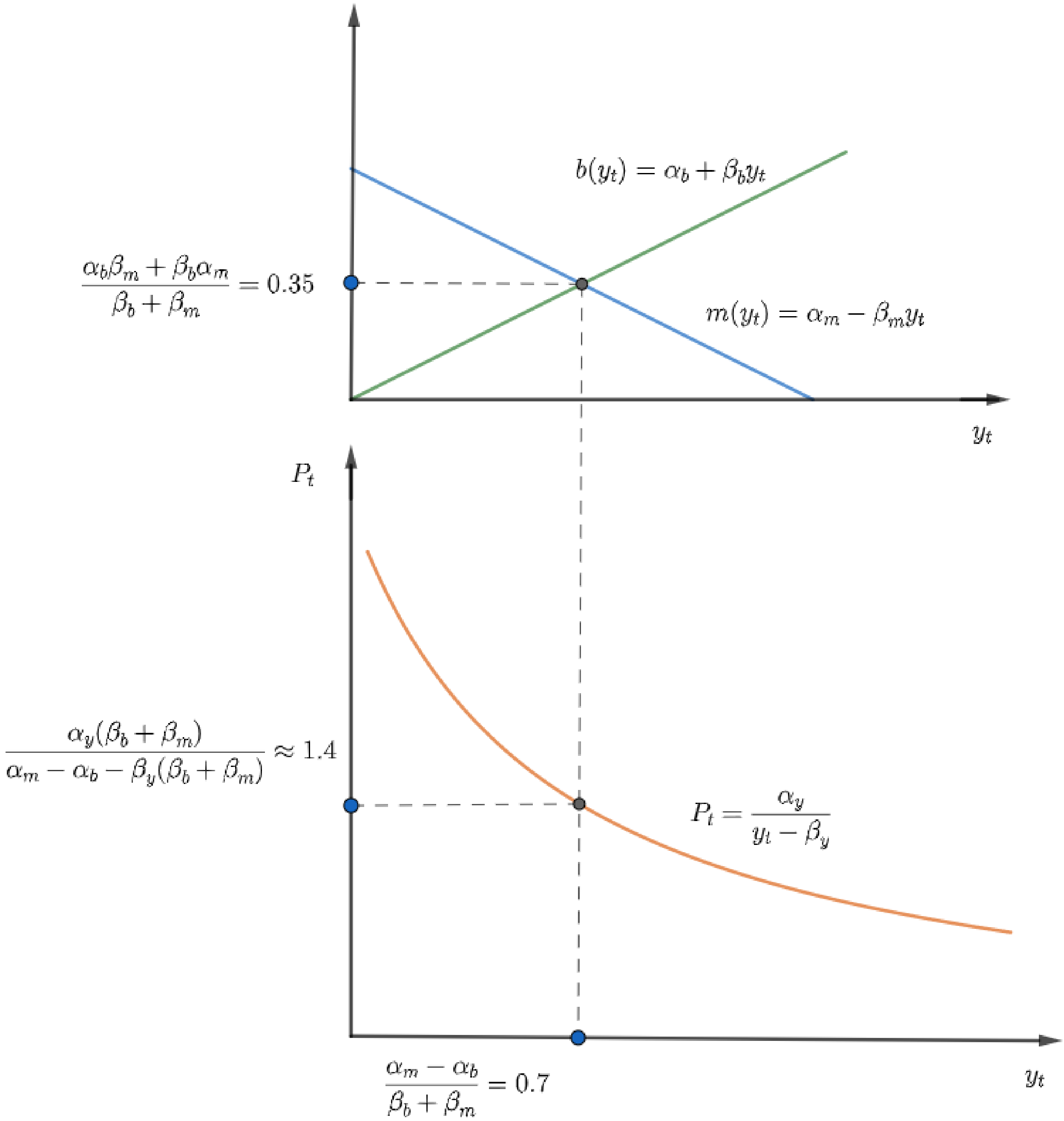


Figure 2: Steady state for $\alpha_b = \beta_y = 0$, $\beta_b = \beta_m = 0.5$, $\alpha_y = 1$ and $\alpha_m = 0.7$. Here $Y^* = \alpha_y + \beta_y P^* = 1$.

Question: Can you guess what happens if $\alpha_m - \alpha_b < 0$?

TD9

2 The Malthusian Regime

---

d. For the rest of the exercise, we assume $\alpha_b = \beta_y = 0$, $\beta_b = \beta_m = 0.5$, $\alpha_y = 1$ and $\alpha_m = \alpha > 0$. What are the steady-state levels for this configuration of parameters?

I more or less computed them in the graph, but I assumed a specific value for α, let's do it again. We have $Y^* = \alpha_y + \beta_y P^* = 1 + 0P^* = 1$. As for y^* and P^*:

$$y^* = \frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} = \frac{\alpha}{1} = \alpha$$

$$P^* = \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} = \frac{1}{\alpha}$$

So $P^* = \frac{1}{\alpha}$, $y^* = \alpha$ and $Y^* = 1$.

e. Show that the model dynamics can be summarized by a first-order difference equation in P_t (of the type $P_{t+1} = f(P_t)$, with f some function that you need to find; you can also look for an equation of the type $\Delta P_t = g(P_t)$ with g some function to find, if it is easier for you to do so).

This question is a very involved way of asking: what are the time dynamics of P_t? You know from your lecture notes that $\dot{P} = [b(y_t) - m(y_t)]P_t$. However, we are in discrete time here, as the question asks for a difference equation (not differential), therefore in this case $\dot{P}$ is substituted by $P_{t+1} - P_t$. We just have to work out the expression above and plug values for the parameters.

$$\begin{aligned}
P_{t+1} - P_t &= [b(y_t) - m(y_t)]P_t \\
&= \left[\cancel{\alpha_b} + \beta_b y_t - \alpha_m + \beta_m y_t\right] P_t && \text{since } \alpha_b = 0 \\
&= [(\beta_b + \beta_m) y_t - \alpha] P_t && \text{since } \alpha_m = \alpha \\
&= [y_t - \alpha] P_t && \text{since } \beta_b + \beta_m = 1 \\
&= \left[\frac{\alpha_y}{P_t} + b_y - \alpha\right] P_t && \text{substituting } y_t(P_t) \\
&= \left[\frac{1}{P_t} - \alpha\right] P_t && \text{since } \alpha_y = 1 \text{ and } \beta_y = 0 \\
P_{t+1} - P_t &= 1 - \alpha P_t \\
P_{t+1} &= P_t(1 - \alpha) + 1
\end{aligned}$$

f. Study the convergence of population to its steady state starting from an initial value of population P_0 close to 0 for the following values of α: (i) $0 < \alpha < 1$, (ii) $\alpha = 1$, (iii) $1 < \alpha < 2$.

This question basically asks you to study the dynamics of population for different values of α. It is more or less about plugging numbers. Let's start from $t = 1$ and see what the dynamics look like. Since $1 - \alpha$ is a bit uncomfortable I substitute it with $\gamma = 1 - \alpha$. Let's start easy and substitute numbers time by time.

$$\begin{aligned} P_1 &= \gamma P_0 + 1 \\ P_2 &= \gamma P_1 + 1 \\ &= (\gamma P_0 + 1)\gamma + 1 \\ &= \gamma^2 P_0 + \gamma + 1 \\ P_3 &= \gamma P_2 + 1 \\ &= (\gamma^2 P_0 + \gamma + 1)\gamma + 1 \\ &= \gamma^3 P_0 + \gamma^2 + \gamma + 1 \end{aligned}$$

You see the pattern. By thinking a little bit you should realise that we can express P_t in the following way:

$$P_t(\gamma) = \gamma^t P_0 + \sum_{s=0}^{t-1} \gamma^s$$

For $t \to \infty$, by the rules of power series, we have:

$$\begin{aligned} P_\infty(\gamma) &= \gamma^\infty P_0 + \sum_{s=0}^{\infty} \gamma^s \\ &= \gamma^\infty P_0 + \frac{1}{1-\gamma} \end{aligned}$$

We are ready to evaluate the convergence. The following table gives a relationship between $1 - \alpha$ and γ .

	α	γ
(i)	$0 < \alpha < 1$	$0 < \gamma < 1$
(ii)	1	0
(iii)	$1 < \alpha < 2$	$-1 < \gamma < 0$

Case (ii) is the easiest. If $\gamma = 0$ then $P_t = 1$ for any t. Population is fixed since the beginning, so in some sense we already converged from the start to 1.

In case (i) we have $0 < \gamma < 1$. If we have no clue we can take one number and see what happens. Let's try $\gamma = 0.5$. We have the following series (assuming P_0 is close to 0):

$$
\begin{aligned}
P_1 &= 0.5P_0 + 1 \approx 1 \\
P_2 &= 0.5(1) + 1 = 1.5 \\
P_3 &= 0.5(1.5) + 1 = 1.75 \\
P_4 &= 0.5(1.75) + 1 = 1.875 \\
P_5 &= 1.9375 \\
P_6 &= 1.96875 \\
&\vdots
\end{aligned}
$$

In the following picture you can see the series graphically:

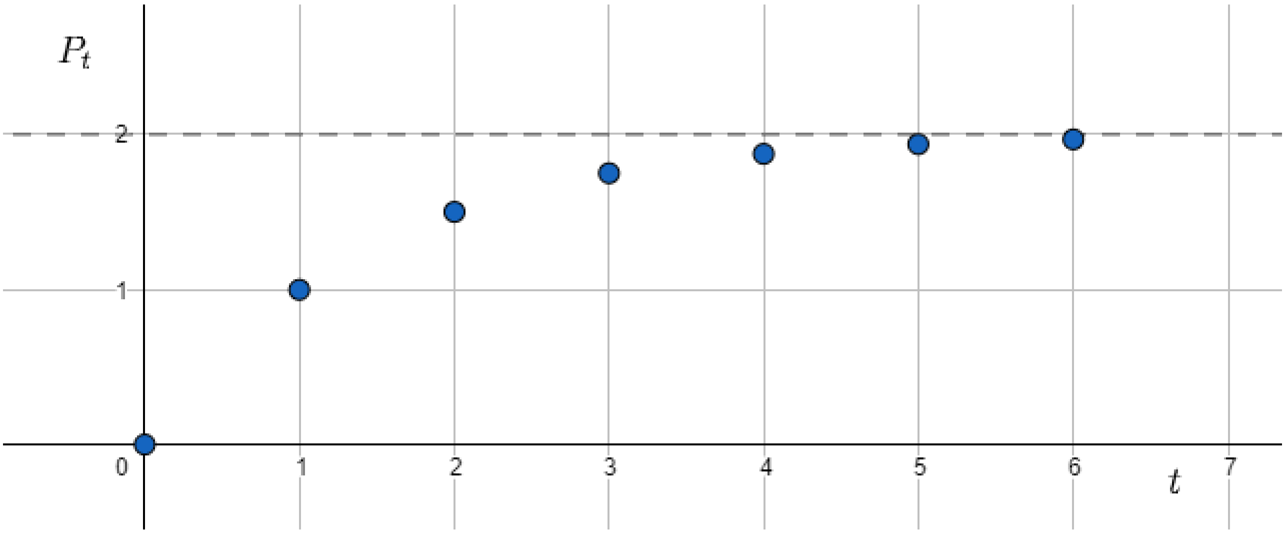


Figure 1: Series with $\gamma = 0.5$.

You can see where we are going. We can immediately compute P_∞ from the expression above (remember that for any $-1 < \gamma < 1$ we have that $\gamma^\infty = 0$):

$$
P_\infty(0.5) = 0P_0 + \frac{1}{1-0.5} = \frac{1}{0.5} = 2
$$

What we conclude is that for a value of α between 0 and 1 population grows, slower at each step, and eventually reaches a steady state level (different for different values of γ).

As for case (iii) , with $-1 < \gamma < 0$, we use the same strategy, namely plugging numbers for the specific value $\gamma = -0.5$. The series looks like this:

$$
\begin{aligned}
P_1 &= -0.5P_0 + 1 \approx 1 \\
P_2 &= -0.5(1) + 1 = 0.5 \\
P_3 &= -0.5(0.5) + 1 = 0.75 \\
P_4 &= -0.5(0.75) + 1 = 0.625 \\
P_5 &= 0.6875 \\
P_6 &= 0.65625 \\
P_7 &= 0.671875 \\
&\vdots
\end{aligned}
$$

As before, I plotted the series:

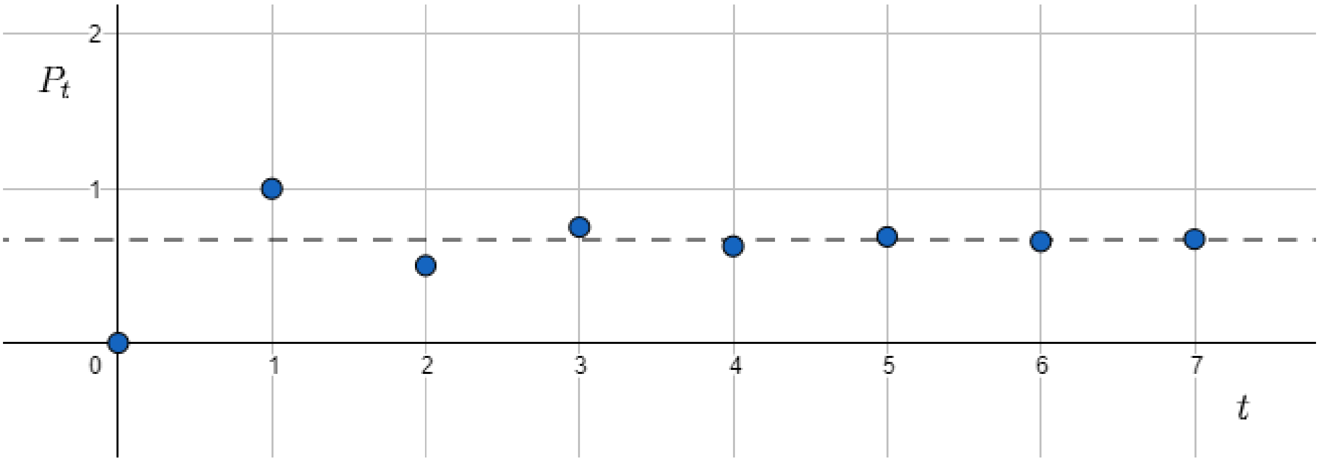


Figure 2: Series with $\gamma = -0.5$.

As you can see the series here goes up and down, it is not monotonic in its growth, contrary to the previous case. However, we can see where it converges to:

$$P_\infty(-0.5) = 0P_0 + \frac{1}{1-(-0.5)} = \frac{1}{1.5} = 0.\bar{6}$$

Interestingly, notice that the term P_0 has in both case no role in determining the convergence, only shaped by γ.

A Solarian Malthus Model

This exercise is set in the world of Solaria, a fictional planet in the Novel "The Naked Sun" by Isaac Asimov. In Solaria few people live without much interaction with each other. The birth rate is completely controlled by the government, as newborns are bred in vitro. Each inhabitant owns a massive amount of land and the space per capita is quite high. Both the land and the house of citizens are completely administered by robots so that the product per capita is huge without the need for a high increase in population. If you want to know more, check the links I provided or read the book! This exercise is a study of the economy of Solaria via the lens of the Malthus Model seen in class.

Assume there are two types of inhabitants in Solaria. The first kind likes robots a lot, while the second a little bit less. The utility function of a generic citizen i is $u_i\left(R_i\right) = \gamma_i \cdot log\left(R_i\right) - cR_i$, where γ_i measures taste for robots and c is the cost of buying a new one. People who like robots a lot are indexed with $i = h$ and they have $\gamma_i = \gamma_h$. Instead, does who do not like robots as much are $i = \ell$ and have $\gamma_i = \gamma_\ell < \gamma_h$.

a. What is the optimal amount of robots for each citizen i? Interpret your result.

Each citizen chooses the amount of robots that maximise his utility function. The problem is the following.

$$\underset{R_i \geq 0}{max} \;\; u_i\left(R_i\right) = \gamma_i \cdot log\left(R_i\right) - cR_i$$

As usual, we take the derivative and set it equal to 0 (what about second order conditions?).

$$\begin{aligned} \frac{\partial\, u_i\left(R_i\right)}{\partial\, R_i} &= \frac{\gamma_i}{R_i} - c = 0 \\ &\Rightarrow R_i^* = \frac{\gamma_i}{c} \end{aligned}$$

The interpretation is quite straightforward. The higher the tastes for robots γ_i, the higher the amount of robots. On the other hands, if the cost c increases then citizen i will buy less robots.

b. A fraction λ of the population likes robots a lot, while the rest $(1-\lambda)$ not that much. If the level of population is P, how many robots R are there in total? How do you interpret $\overline{\lambda} = \lambda\gamma_h + (1-\lambda)\,\gamma_\ell$?

The share λ of the population will buy an amount $R_h^* = \frac{\gamma_h}{c}$ of robots, while $(1-\lambda)$ will get $R_\ell^* = \frac{\gamma_\ell}{c}$. If the population is P, then the total amount of robots will be

$$\begin{aligned} R &= \lambda P \frac{\gamma_h}{c} + (1-\lambda)\,P\frac{\gamma_\ell}{c} \\ &= \frac{P}{c}\left(\lambda\gamma_h + (1-\lambda)\,\gamma_\ell\right) \\ &= \frac{P}{c}\overline{\lambda} \end{aligned}$$

The variable $\overline{\lambda}$ captures the average tastes for robots in the population. The higher the λ, γ_h or γ_ℓ, the higher the citizens of Solaria like robots on average.

c. Production is given by the number of robots in the economy. Assume that production is linear in robots $Y(R) = \alpha_y + \beta_y R$. Write it as a function of population explicitly. What is production per capita y?

This is just a matter of substitution.

$$\begin{aligned} Y(R) &= \alpha_y + \beta_y R \\ &= \alpha_y + \beta_y \frac{P}{c}\left(\lambda\gamma_h + (1-\lambda)\,\gamma_\ell\right) \\ Y(P) &= \alpha_y + \beta_y \frac{P}{c}\overline{\lambda} \end{aligned}$$

As usual, production per capita is $y = \frac{Y}{P}$.

$$y(P) = \frac{Y(P)}{P} = \frac{\alpha_y}{P} + \beta_y \frac{\overline{\lambda}}{c}$$

This is the first step for constructing a Malthus model from scratch, we miss the law of motion of population, which is the state variable.

d. I mentioned that the birth rate is controlled, how do you translate this assumption? What is $b(y)$?

Since the amount of newborns is fully controlled by the government and does not depend on production, it is equal to a constant $b(y) = k$ where k is a constant.

<u>*Question:*</u> Do you have any ideas about real circumstances in which the same is true?

e. From now on assume that $c = 1$. The mortality rate is linear in production per capita $m(y) = \alpha - \beta y$. You have all the ingredients. Find the steady-state level of production per capita and population. Represent everything in a nice graph.

As usual, the law of motion of population is $\dot{P}_t = [b - m(y)]\,P_t$. Hence, in steady-state we need $\dot{P} = 0$, which means $P = 0$ or $b = m(y)$. The second condition implies the following.

$$
\begin{aligned}
k &= \alpha - \beta y \\
\Rightarrow \beta y &= \alpha - k \\
\Rightarrow y^* &= \frac{\alpha - k}{\beta}
\end{aligned}
$$

As for population, we can find its steady-state level by exploting the relation between P and y. By substituting y^*we obtain:

$$
\begin{aligned}
y^* &= \frac{\alpha_y}{P*} + \beta_y \overline{\lambda} \\
\frac{\alpha - k}{\beta} &= \frac{\alpha_y}{P*} + \beta_y \overline{\lambda} \\
\frac{\alpha - k}{\beta} - \beta_y \overline{\lambda} &= \frac{\alpha_y}{P*} \\
P^* \left[\frac{\alpha - k}{\beta} - \beta_y \overline{\lambda}\right] &= \alpha_y \\
P^* \left[\frac{\alpha - k - \beta_y \beta \overline{\lambda}}{\beta}\right] &= \alpha_y \\
P^* &= \frac{\alpha_y \beta}{\alpha - k - \beta_y \beta \overline{\lambda}}
\end{aligned}
$$

To draw the curve of P as a function of y

we employ their relation found previously.

$$\begin{aligned} y(P) &= \frac{\alpha_y}{P} + \beta_y \overline{\lambda} \\ y - \beta_y \overline{\lambda} &= \frac{\alpha_y}{P} \\ P\left[y - \beta_y \overline{\lambda}\right] &= \alpha_y \\ P &= \frac{\alpha_y}{y - \beta_y \overline{\lambda}} \end{aligned}$$

We can now put the model to use and see if we can learn something about Solaria.

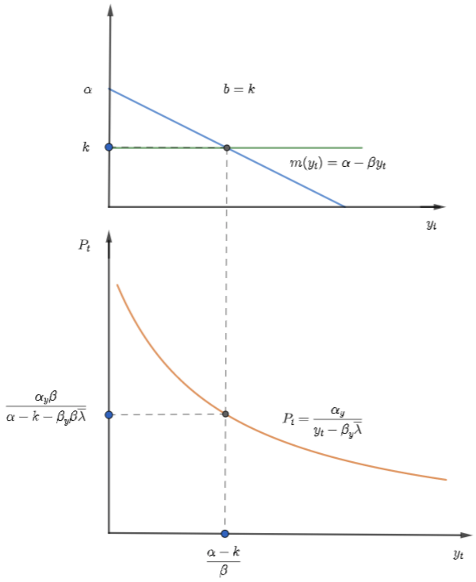


Figure 1: Steady-state level of population and production per capita.

f. How does the steady-state level of population depends on $\overline{\lambda}$ or, λ and γ_h, γ_ℓ? Try to simulate a positive shock to $\overline{\lambda}$ and see what happens in the steady state.

The only curve that is affected by an increase in $\overline{\lambda}$ is P as a function of y. The following picture represents the change. In the new steady state y is unaffected, while P increases. All the production of the increased number of robots is eaten by new population, and resources per capita stay the same. Is this what is going on in Solaria? Maybe we have to look at something else.

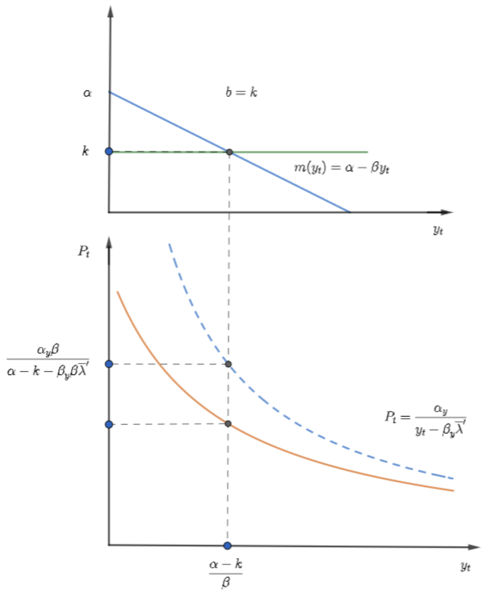


Figure 2: Positive shock to $\overline{\lambda}$.

g. What happens if the government decides to lower the birth rate? What do you think the birth rate and $\overline{\lambda}$ are in Solaria? Any comments?

The only curve affected by a decrease in k is the birth rate. The new situation is depicted in the figure here. The lower number of births leads to a decrease in population, and hence those who remains enjoy more resources per capita. This is more or less what happens in Solaria, so maybe this channel is the relevant one to study its economy. Or maybe this is the wrong model to tackle the problem 🙂. What do you think?

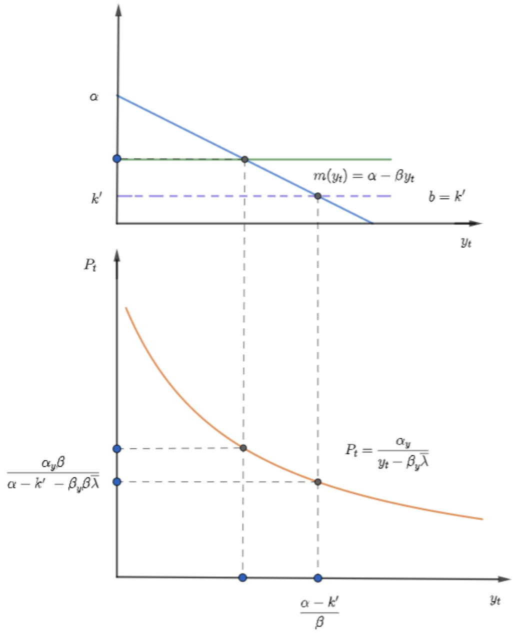


Figure 3: Negative shock to k.

Baby boom (from midterm 2019)

This exercise is not significantly different to exercises 2 and 3 of TD1! You should already know how to do the computations and the graph of the first questions. The second part is a little bit more difficult and it requires you to translate what is happening in mathematics. We have $F(k_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$. Investment is, as always, $I = sY = sF(K_t, L_t)$. The law of motion of capital is $\Delta K_t = K_{t+1} - K_t = I - \delta K_t$. The work population grows at rate $n = \frac{L_{t+1} - L_t}{L_t} = 0.05$. We also have that $\delta = 0.05,\ s = 0.1$ and $\alpha = 0.5$.

Remark: In these solutions I spell out every step just for you to understand, as you can see from the professor's solutions it is not needed that you specify each step as I do. However, for sure it can not hurt you.

a. On the balanced growth path, k and y are stable. Compute their numerical values.

The answer to this question requires no more than performing the computations we have been doing in the previous TDs and substituting numbers. First, we have to convert everything in per capita terms. The production function becomes:

$$\begin{aligned} \frac{1}{L_t} F(K_t, L_t) &= F\left(\frac{K_t}{L_t}, \frac{L_t}{L_t}\right) \\ &= \left(\frac{K_t}{L_t}\right)^{\alpha} \left(\frac{L_t}{L_t}\right)^{1-\alpha} \\ f(k_t) &= k_t^{\alpha} \end{aligned}$$

You should recognise that we have been using this function a lot! Is the classical Cobb-Douglas. To compute the numerical value of k in the balanced growth path we can rely on the condition under which this variable is indeed on such path, which means that its growth rate is equal to zero. Hence, we must first compute its growth rate.

$$
\begin{aligned}
\frac{\Delta k_t}{k_t} &= \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} \\
&= \frac{sF(K_t, L_t) - \delta K_t}{K_t} - n \\
&= \frac{s\frac{1}{L_t}F(K_t, L_t) - \delta \frac{1}{L_t}K_t}{\frac{1}{L_t}K_t} - n \\
\frac{\Delta k_t}{k_t} &= \frac{sf(k_t) - \delta k_t}{k_t} - n \\
\Delta k_t &= sf(k_t) - \delta k_t - nk_t \\
&= sf(k_t) - (\delta + n)k_t
\end{aligned}
$$

The condition for being in steady state its growth rate equal to 0, thus, we check what this condition implies in this exercise.

$$\Delta k_t = 0 \Leftrightarrow sf(k_t^*) - (\delta + n)k_t^* = 0 \Leftrightarrow s(k^*)^\alpha = (\delta + n)k_t^*$$

By substituting the numbers we are given in the text we obtain that:

$$k^* = \left(\frac{\delta + n}{s}\right)^{\frac{1}{\alpha - 1}} = \left(\frac{0.05 + 0.05}{0.1}\right)^{-2} = 1$$

Since $y^* = f(k^*)$ we obtain:

$$y^* = (k^*)^\alpha = 1^{1/2} = 1$$

b. Is the savings rate s is at its golden rule value? If not, what should be the golden-rule savings rate?

From problem 2 of TD 1 you may recall that the golden rule savings rate is $s = \alpha = 0.5$, while in this case we have that $s = 0.1 \neq 0.5$! However, let's try to prove it again. There are many ways to do it. The first one is to express consumption as $c^* = (1 - s)f(k^*)$ and compute the s that maximises it in steady state. The steps to do this are detailed in the solution of TD 1. A different method could be to check the k which comes from the golden rule of savings and derive the s for which we get that k. The steps are the following. First, express consumption only as a function of capital:

$$
\begin{aligned}
c^* &= (1-s)y^* \\
&= (1-s)f(k^*) \\
&= f(k^*) - sf(k^*) \\
&= f(k^*) - (\delta+n)k^*
\end{aligned}
$$

Where the last step come from the steady state condition we derived in the previous point. We then obtain the k that maximises consumption by solving a maximisation problem:

$$
\begin{aligned}
\frac{\partial c^*}{\partial k^*} = 0 &\Rightarrow f'(k^*) - (\delta+n) = 0 \\
&\Rightarrow \alpha(k^*)^{\alpha-1} = (\delta+n) \\
&\Rightarrow k^* = \left(\frac{\delta+n}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
&\Rightarrow k^* = \left(\frac{0.05+0.05}{0.5}\right)^{\frac{1}{0.5-1}} = 25
\end{aligned}
$$

Which is quite different from the $k^* = 1$ we got before! What is the s that rationalises this result? From the expression we derived in the previous point we have:

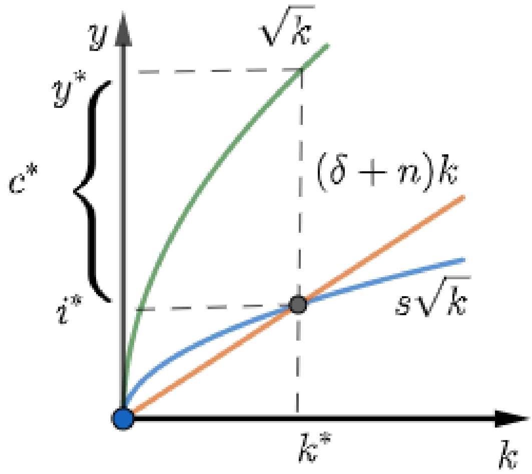
$$
\begin{aligned}
k^* &= \left(\frac{\delta+n}{s}\right)^{\frac{1}{\alpha-1}} \\
25 &= \left(\frac{0.05+0.05}{s}\right)^{\frac{1}{0.5-1}} \\
25 &= \left(\frac{0.1}{s}\right)^{-2} \\
25 &= \left(\frac{s}{0.1}\right)^{2} \\
5 &= \left(\frac{s}{0.1}\right) \\
0.5 &= s \neq 0.1
\end{aligned}
$$

Which indeed gives us $\alpha = s$. Remember that $i^* = sy^* = 0.1(1) = 0.1$, moreover $c^* = y^* - i^* = 1 - 0.1 = 0.9$.

c. Make a plot like the ones made in class with k on the horizontal axis and y on the vertical axis. Draw $f(k) = k^{0.5}$. Draw $sf(k)$. Draw $(n+\delta)k$. Indicate k^* and y^*.

The graph is ecxactly the one we did in exrcise 3 odf TD 1! I'll report the same picutre I put there. The production function is $f(k_t) = (k_t)^{\frac{1}{2}} = \sqrt{k_t}$, while $\delta = n = 0.05, y^* = 1$ and $k^* = 1$ (you may want to substitute the numbers in your graph).

1. $\sqrt{k}$
2. $(\delta + n)k_t$
3. $s\sqrt{k}$



Graph from exercise 3 TD1.

d. After World War 2, many American soldiers fighting in Europe came back home and made lots of babies. Imagine that during World War 2, the American economy is on the balanced growth path. Then when troops come home at time t, the population L doubles. What is the numerical value of the growth rate of k_t just after the doubling? (If you don't have a calculator, you can leave a mathematical expression as is (as long as it just involves numbers, no variables).)

We are asked to compute the growth rate of k_t at time t, when the populatoin doubles. We have to perform the exact computations we did in the previous points, but instead of having L_t, we have $2L_t$. We start from the production function:

$$
\begin{aligned}
\frac{1}{2L_t}F(K_t, 2L_t) &= F\left(\frac{K_t}{2L_t}, \frac{2L_t}{2L_t}\right) \\
&= \left(\frac{K_t}{2L_t}\right)^{\alpha}\left(\frac{2L_t}{2L_t}\right)^{1-\alpha} \\
f(k_t) &= \left(\frac{k_t}{2}\right)^{\alpha} = \bar{k}_t^{\alpha}
\end{aligned}
$$

Now we have to compute the growth rate. Again the calculations follow the same logic as before:

$$
\begin{aligned}
\frac{\Delta k_t}{\frac{k_t}{2}} &= \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} \\
&= \frac{sF(K_t, 2L_t) - \delta K_t}{K_t} - n \\
&= \frac{s\frac{1}{2L_t}F(K_t, 2L_t) - \delta \frac{1}{2L_t}K_t}{\frac{1}{2L_t}K_t} - n \\
\frac{\Delta k_t}{\frac{k_t}{2}} &= \frac{sf(k_t) - \delta \frac{k_t}{2}}{\frac{k_t}{2}} - n \\
\Delta k_t &= sf(k_t) - \delta\frac{k_t}{2} - n\frac{k_t}{2} \\
&= sf(k_t) - (\delta + n)\frac{k_t}{2}
\end{aligned}
$$

By expliciting the production function we obtain:

$$
\Delta k_t = s\left(\frac{k_t}{2}\right)^{\alpha} - (\delta + n)\frac{k_t}{2} = 0.1\left(\frac{1}{2}\right)^{0.5} - (0.05 + 0.05)\frac{1}{2} = 0.0207
$$

However, we must find the growth rate, not only the Δ:

$$
\frac{\Delta k_t}{\frac{k_t}{2}} = \frac{0.0207}{\frac{1}{2}} = 0.0414
$$

e. As previously, imagine that during World War 2, the American economy is on the balanced growth path. Then when troops come home at time t, the population L doubles. Also, since these young people want to make babies as fast as possible, at the

same moment, the growth rate jumps from $n = 0.05$ to $n = 0.15$. Reproduce the graph in c. and show how the long run equilibrium will change. Indicate what happens to k just after soldiers get back (at time t) and where the economy converges in the long run (indicate numerical values).

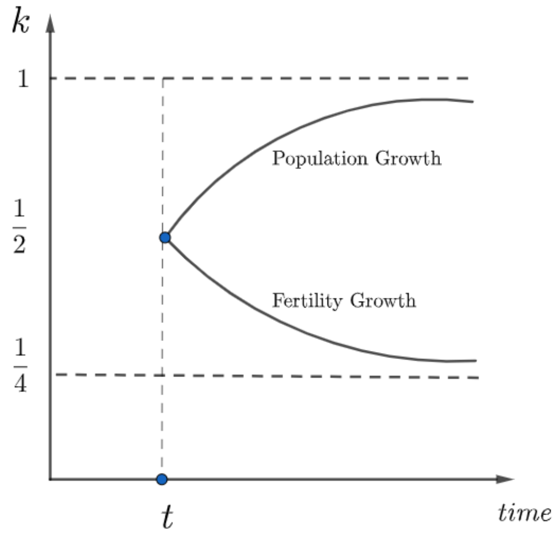
Before showing the graph we need to compute the new k^* and y^*. As always, we set the law of motion equal to zero:

$$\begin{aligned}\Delta k_t &= 0\\ &\Leftrightarrow s\left(\frac{k_t^*}{2}\right)^\alpha - (\delta+n)\frac{k_t^*}{2} = 0\\ &\Leftrightarrow \left(\frac{k_t^*}{2}\right)^{\alpha-1} = \frac{\delta+n}{s}\\ &\Leftrightarrow \left(\frac{k_t^*}{2}\right) = \left(\frac{0.05+0.015}{0.1}\right)^{\frac{1}{0.5-1}} = \frac{1}{4}\end{aligned}$$

As for y^* we have that $y^* = f(k^*) = \left(\frac{k_t^*}{2}\right)^{0.5} = \left(\frac{1}{4}\right)^{0.5} = \frac{1}{2}$. I do not include the graph here again as it is the same graph as before with different numbers.

f. Plot k over time around the period t (hence make a graph with time on the horizontal axis and k on the vertical axis). On the same graph, show how k adjusts after t in the situation described in d. and in the situation described in e. Make sure to show where k is converging in each case.

At time t the input in the production function is immediately halved due to the increase in population. If there is no fertility growth after a while k will return on its original balanced growth path. In the case of fertility growth, instead, it will converge to $\frac{1}{4}$.



g. In the context of this model, are American workers (those always in the US) better off before time t (before the influx of workers), some time after time t if fertility does not change d. some time after t or if fertility does change e.? Rank the three situations from best to worst and justify briefly.

This question just amounts to compare the y^* in different circumstances. In $\tau < t$ we have $y^*_\tau = 1$, when fertility increase $(n = 0.15)$ occurs at time t we have that the new steady state gdp is $y^* = \frac{1}{2}$, while without the increase in n but after the increase in population we are outside the growth path leading to $y^* = 1$, and therefore we are slightly below this value. Therefore, the best situation is the one before the shock, then we have the increase in population without the increase in n and lastly the worse situation is when we also observe a fertility rate increase.

Midterm 2020

Remark: Some solutions here are way more elaborate then what was needed to get full points! This is just to make you understand.

2 - Production in the Solow-Swan model

Here we have a Solow-Swan model with population growth $n \in (0,1)$, no technological growth, where $s \in (0,1)$ is the saving rate, K is the stock of capital, $k = \frac{K}{L}$ is capital per worker, $F(K,L)$ is the production function with the usual assumptions, $f(k) = \frac{F(K,L)}{L}$ is production per worker, $\delta \in (0,1)$ is the rate of capital depreciation, and finally k^* denotes k on the balanced growth path.

In this exercise I put both the general answer and the particular case in which $f(k) = k^\alpha$ to show you that the solutions make sense and to make you visualise them better.

Is $s\frac{\delta f(k)}{\delta k} - \delta - n$ greater than zero, equal to zero or smaller than zero or we cannot say without more information?

▼ **1.** *If $k \to 0$.* In this case $\frac{\delta f(k)}{\delta k} \to \infty$ due to the Inada condition, therefore the left side of the equation is way bigger then the right side and therefore the expression is positive.

Example: In the special Cobb-Douglas case $\frac{\delta k^\alpha}{\delta k} = \alpha k^{\alpha-1} = \frac{\alpha}{k^{1-\alpha}}$. Since k is at the denominator you can clearly see that if it goes to 0 then the ratio goes to infinity.

▼ **2.** *If $k = k^*$.* A lot of students got this wrong. The standard logic I saw was that since $k = k^*$ we are on the balanced growth path and therefore $sf(k^*) = (\delta - n)k^*$, hence $sf(k^*)' = (\delta + n)$. However, this logic is fallacious. If two functions are equal in one point, which in our case is k^*, then it is not true that their derivative is equal in that point in general. In fact, it is true only if those two functions are equal in every point. This is kind of a technical argument, I understand it is not straightforward, but maybe you will be convinced by the solution. Remember that the derivative of a function tells us how much that function varies after an infinitesimal change in the variable you take the derivative with respect to. Moreover, you know two things. For k a little bit smaller than k^*, say k_- we have that $sf(k_-) > (\delta + n)k_-$ (you can see it from the standard graph). Also, if k is a little bit higher than k^*, call it k_+ it holds that $sf(k_+) < (\delta + n)k_+$. This means that in the process of going from k_- to k_+ the function $(\delta + n)k$ had a higher increase then $sf(k)$, and therefore its derivative $(\delta + n)$ is higher then the derivative $sf'(k)$. Since this holds for all $k_- < k^* < k_+$ then it also holds for k^*.

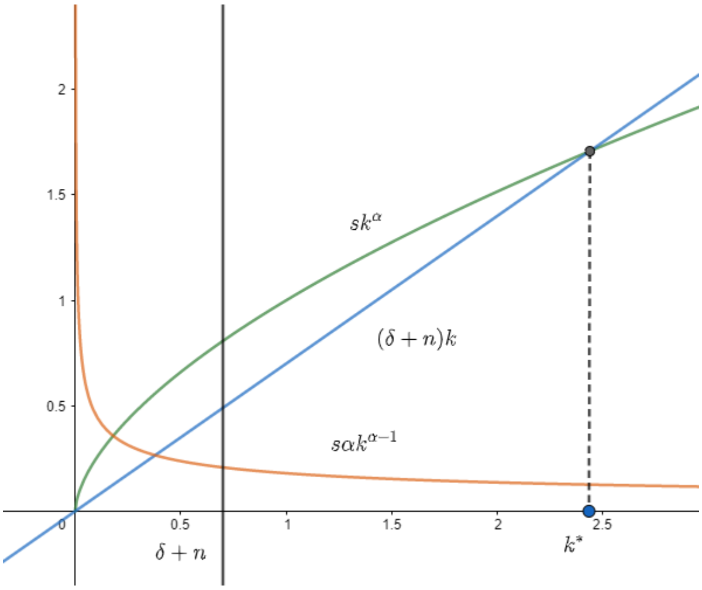


Figure 1: Graph with $f(k) = k^{\alpha}$, $s = 0.3$, $\alpha = 0.6$, $\delta = 0.3$, $n = 0.4$.

▼ **3.** *If $k \to \infty$*. This is the converse of the point we had before. Due to the Inada conditions $\frac{\delta f(k)}{\delta k} \to 0$ and therefore only the right (negative) side of the equation is left.

<u>*Example*</u>: As before, you can check that $\frac{\alpha}{k^{1-\alpha}} \to 0$ and therefore the general reasoning holds also here.

▼ **4.** *Is $\frac{\delta f(k)}{\delta k} - \delta - n$ greater than zero, equal to zero or smaller than zero or we cannot say without more information if $k = k^*$? (Note the difference is that here there is no s in the expression.).* I saw two different kind of mistakes in this question. The first one was to assume that we were in the golden rule and therefore $\frac{\delta f(k^*)}{\delta k} = \delta - n$. However, this was not specified by the question. We could have any $s \in (0,1)$. The other typical mistake was to say that since $s\frac{\delta f(k^*)}{\delta k} = \delta - n$ and $s < 1$ then if you remove it you get $\frac{\delta f(k^*)}{\delta k} > \delta - n$. This logic is faulty, as it does not consider that $\frac{\delta f(k^*)}{\delta k}$ itself depends on s, and therefore it is different for different values of s. Hence, we can not determine the relationship between the two quantities.

<u>*Example*</u>: In the Cobb-Douglas case we have that $k^* = \left(\frac{\delta+n}{s}\right)^{\frac{1}{1-\alpha}}$, therefore $f'(k^*) = \alpha\left(\frac{\delta+n}{s}\right)^{\frac{1-\alpha}{1-\alpha}} = \alpha\left(\frac{\delta+n}{s}\right)$, therefore the question becomes: what is the sign of $\alpha\left(\frac{\delta+n}{s}\right) - \delta - n$? As an example:

$$\alpha\left(\frac{\delta+n}{s}\right) - \delta - n > 0$$
$$\alpha\left(\frac{\delta+n}{s}\right) > \delta + n$$
$$\frac{\alpha}{s} > 1$$
$$\alpha > s$$

3 - A general purpose technology

Consider the production function with general purpose technology $Y = F(A, K, L) = AK^{\alpha}L^{1-\alpha}$ where $\alpha \in (0, 1)$, K is the capital stock and L is the labor. Assume that $A_{t+1} = (1 + g)A_t$ and $L_{t+1} = (1 + n)L_t$ where g and n are exogenous constants. Capital depreciates at rate δ. You can use the same approximation as in the TD for the growth of some variable X: $g_X = log\left(\frac{X_{t+1}}{X_t}\right)$.

This exercise consists in playing with growth rates, the techniques are the same we used in the TDs.

▼ **1.** *If the growth rate of capital at time t is $g_{K,t}$, compute the growth rate of Y at time t in terms of $g_{K,t}$ and the growth rates of technology and labour.* Most if you got this question right. The idea is to exploit the definition given in the text and do the calculations we did in the TDs. Some of you thought that the α exponent was also on A, probably it was just a misreading of the text. Also, some students took derivatives with respect to time, but here time is discrete!

$$\begin{aligned} g_{Y,t} &\approx log\left(\frac{Y_{t+1}}{Y_t}\right) \\ &= log\left(\frac{A_{t+1}K_{t+1}^{\alpha}L_{t+1}^{1-\alpha}}{A_tK_t^{\alpha}L_t^{1-\alpha}}\right) \\ &= log\left(\frac{A_{t+1}}{A_t}\right) + \alpha log\left(\frac{K_{t+1}}{K_t}\right) + (1-\alpha)log\left(\frac{L_{t+1}}{L_t}\right) \\ &= g + \alpha g_{K,t} + (1-\alpha)n \end{aligned}$$

▼ **2.** *Let's now define general-purpose technology-adjusted labor as $\bar{L} = A^{\frac{1}{1-\alpha}}L$. What is the growth rate of $\bar{L}$?* The procedure for answering this question is the same as before, we just need to use the expression of $\bar{L}$. Most of you got this correct.

$$\begin{aligned} g_{\bar{L}} &= log\left(\frac{A_{t+1}^{\frac{1}{1-\alpha}}L_{t+1}}{A_t^{\frac{1}{1-\alpha}}L_t}\right) \\ &= \frac{1}{1-\alpha}log\left(\frac{A_{t+1}}{A_t}\right) + log\left(\frac{L_{t+1}}{L_t}\right) \\ &= \frac{1}{1-\alpha}g + n \end{aligned}$$

▼ **3.** *What will be the growth of output per worker in the long run?* As always, output per worker is $\frac{Y}{L}$, some of you got confused by the $\bar{L}$ and computed the wrong growth rate. So we have (from now on I omit the logarythm transformation):

$$\frac{Y}{L} = \frac{AK^{\alpha}L^{1-\alpha}}{L} = A\left(\frac{K}{L}\right)^{\alpha}$$

$$g_y = g + \alpha(g_{K,t} - n)$$

▼ **4.** *What will be the growth rate of real wages in the long run?* Real wage is nominal wage times units of technology adjusted labour *over number of workers*. Before starting, notice that First, notice that $g_{\bar{A}} = \frac{g}{1-\alpha}$. We can now compute the growth rate:

$$w_r = \frac{wLA^{\frac{1}{1-\alpha}}}{L} = wA^{\frac{1}{1-\alpha}}$$

$$g_{w_r} = g_w + g_{\bar{A}} = \frac{g}{1-\alpha}$$

▼ **5.** *What will be the growth rate of the real interest rate in the long run?* From the lecture notes you know that r is stable, which implies that it's growth rate is 0 (not generally constant!).

5 - Steady-States

In this exercise you had to deal with a strange production function. We briefly talked about it in one of the TDs, but if you did not remember you had to think carefully to get the answer correct.

▼ **1.** *In the Solow-Swan model with constant population and technology, how many steady-states are there in total if $\frac{\delta F(K)}{dK} = c$ where c is a constant?* Here it was key to understand that the production function is linear. Consider $F(K) = a + cK$, then you have $\frac{\partial F(K)}{\partial K} = c$. We already saw what happens if the production function is linear in TD2. First, there is a steady state in 0. Second, there could be other infinitely many steady states if $\delta = sc$, which is in fact the condition for being in a steady state. The answer to this question should be clear from the graph below.

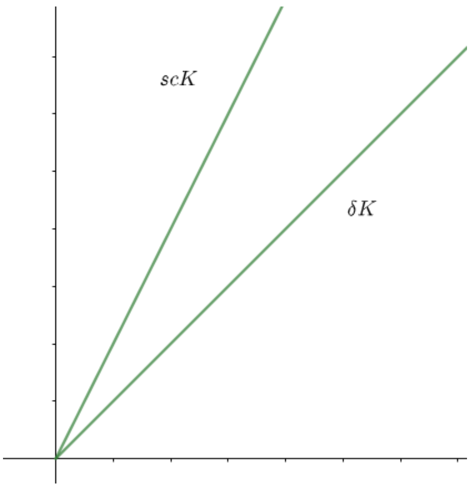


Figure 2: Steady state with linear production function.

▼ **2. (Bonus)** *In the Solow-Swan model with constant population and technology, how many stable steady-states are there in total if $\frac{\delta F(K)}{dK} = c$ where c is a constant? A steady-state is stable if for small deviation around K, the economy returns to K automatically.* This question was a bonus as you did not work with the "stability" concept before. However, is exactly what we did in the previous TD, so you should be a little bit familiar with it by now. Consider the two different cases we studied before. First, it could be that $sc \neq \delta$ and therefore we only have one steady state at 0. Is it stable? What happens if we perturb it a little bit and move on $0 + \epsilon$? It depends. If $sc > \delta$ then there is more investment then depreciation, and K increases more than what is lost due to δ. In this case the steady state is not stable, as we do not return to 0. If instead $sc < \delta$, then after a small increase capital still depreciates at a higher rate then what is saved. Hence, we return back to 0 and the state is stable. In the case $sc = \delta$ any point is a steady state and therefore no state is stable, as if we move a little bit we are already in a new steady state.

6 - An Algal Bloom

Consider the dynamics of the fish population with fishing. At time $t-1$, the stock of fishes is at its long term equilibrium $S^* > 0$. Then, at time t, an algal bloom kills half of the fish population. The bloom lasts a single period. At $t+1$, the algal bloom has ended and the ecosystem is back to where it was before (except that fishes have died of course).

This exercise did not require hard computations, you had to reason about the question and understand more or less intuitively the direction of the answer.

▼ **1.** *At $t+1$, is the net growth of the stock $\Delta S_{t+1} = S_{t+2} - S_{t+1}$ higher, equal or lower than the net growth at $t-2$, or we don't have enough information to tell. Assume that the number of boats have not changed between $t-2$ and $t+1$.* The question here asks to compare ΔS_{t+1} to $\Delta S_{t-2} = S_{t-1} - S_{t-2}$. Since before t we were in steady state, we must have that $\Delta S_{t-2} = 0$ and $S_{t-1} = S_{t-2} = S^*$. Hence, the question becomes: is $\Delta S_{t+1} > 0$? At time t we have the algal bloom, so the steady state population gets halved and we have that $S_t = \frac{S^*}{2}$. At $t+1$, even if the bloom is over, we are not at the steady state as $\frac{S^*}{2} < S^*$. To reach the steady state again the stock of fish must grow positively. For some time periods τ, we will have that $S_{\tau+1} > S_\tau$. We reached the conclusion that $\Delta S_{t+1} > 0$. Not all of you got this correct, I think it may be due to the confusion with the definition of net growth rate.

▼ **2.** *At $t+1$, is the natural growth of the stock $\tau(S_{t+1})$ higher, equal or lower than the natural growth at $t-2$ $\tau(S_{t-2})$, or we don't have enough information to tell. Assume that the number of boats have not changed between $t-2$ and $t+1$.* This question asks to compare the levels of the parabola for different values of S_t. In S_{t-2} we were in steady state, but we do not know where! As you can see from the graph, values of $\tau\left(\frac{S^*}{2}\right)$ could be both higher or lower than $\tau(S^*)$.

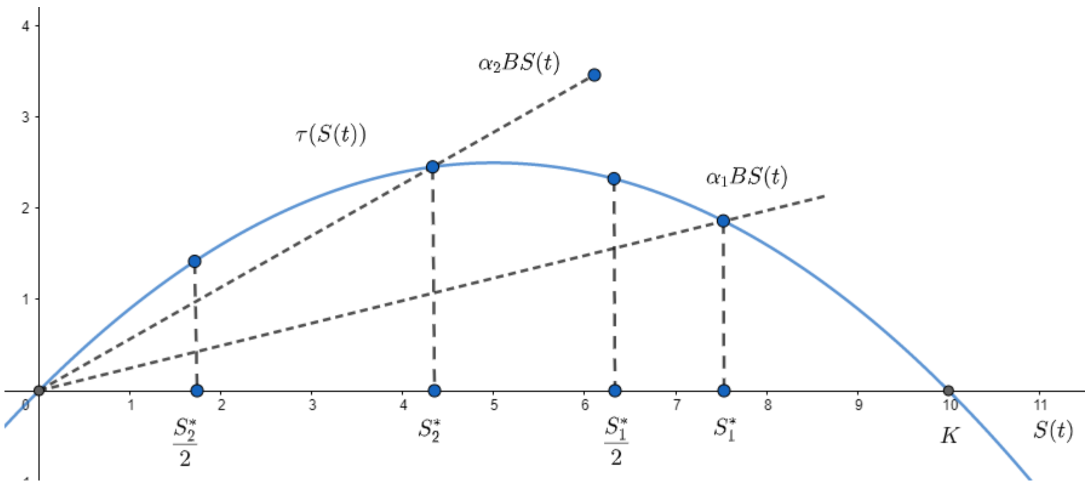


Figure 3: Steady states and growth rate for different fishing intensities.

▼ **3.** *If commercial fishing was very intense, could this bloom cause the population to go extinct?* Here you should consider that fishermans' technology is $\alpha < 1$, therefore they will always be able to catch a fraction of the fish that is lower than the total amount available. If $\alpha \to 1$ there will still be a small fraction ϵ of fishes around. You can see from the graph of the growth rate τ that this is positive at any value greater than 0, so extinction is not possible in this model (does not mean it is not possible in reality!).