Game Theory — Exam - A

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Time: 75 minutes. Be clear and concise.

Part I — Definitions and True/False

Question 1 (4 points)**.** Define an extensive form with perfect information. What do you need to add to get an extensive form **game** with perfect information?

Question 2 (7 points)**.** Say whether the following statements are true, false, or uncertain, and *justify* your answer. ("uncertain" means that the statement is true under conditions that are not explicitly mentioned.)

(a) **[3.5 points]** Consider the following strategic-form game where the two players have expected utility preferences.

	H	T
H	$(1, -1)$	$(-1, 1)$
T	$(-1, 1)$	$(1, -1)$

Statement: "This game has a Nash equilibrium in pure strategies."

(b) **[3.5 points] Statement:** "The following extensive form has a proper subgame, i.e. a subgame other than the whole game."

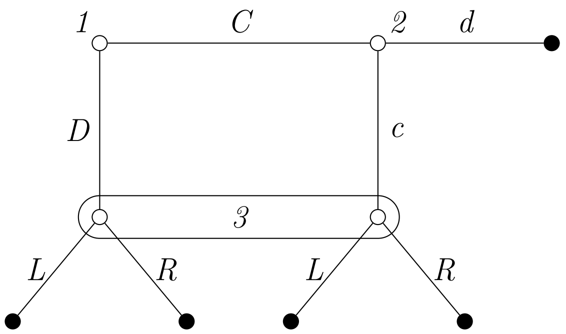

Figure 1: Extensive-form for statement (b).

Part II — Exercises

Question 3 (7 points)**.** Consider the following strategic-form game with two players who have expected utility preferences:

	L	M	R
A	$(5, 2)$	$(1, 0)$	$(2, 0)$
B	$(0, 1)$	$(3, 4)$	$(2, 0)$
C	$(2, 3)$	$(1, 3)$	$(1, 0)$

(a) Perform the Cardinal Iterated Deletion of Strictly Dominated Strategies.

(b) Find all Nash equilibria, both in pure and in mixed strategies of the game.

Question 4 (7 points)**.** An entrant, Ann, first decides whether to stay *Out* of a market or to go *In*. If she stays *Out*, the incumbent, Bob, keeps the market alone and Ann earns 2 while Bob earns 5 of profits. If Ann goes *In*, the two firms must *simultaneously* commit to a technological standard — each picks the format its product will use (say, the charging cable of the product), and they sell only if their choices are compatible. The resulting profits for Ann and Bob are:

	b_1	b_2
a_1	$(4, 1)$	$(0, 0)$
a_2	$(0, 0)$	$(1, 4)$

Table 1: Simultaneous move game of technological standards.

For both firms, payoffs equal profits.

(a) Represent the situation as an extensive-form game with imperfect information. How many subgames does it have?

(b) Find all Nash equilibria, pure and mixed, of the simultaneous game in Table 1.

(c) Find all subgame-perfect equilibria of the game.

(d) Exhibit a Nash equilibrium that is *not* subgame-perfect, and explain why it fails subgame perfection.

Part III — Advanced Question

Question 5 (6 points)**.** Two flatmates simultaneously decide whether to contribute to a shared public good — for example, cleaning their common kitchen. Each player $i \in \{1, 2\}$ chooses $g_i \in \{0, 1\}$: either *Contribute* ($g_i = 1$) or *Not* ($g_i = 0$). Each contribution costs the contributor $c > 0$ and produces a benefit $b > 0$ enjoyed by *both* flatmates. Player i's payoff is

$$\pi_i(g_1, g_2) = b\,(g_1 + g_2) - c\,g_i.$$

Throughout, assume $b < c < 2b$.

(a) Interpret the assumption $b < c < 2b$ in terms of the incentives to contribute.

(b) Write the 2×2 payoff matrix of this game in terms of b and c.

(c) Find all the Nash equilibria of the game.

(d) Is any of the Nash equilibria Pareto efficient? Identify the outcome that both players prefer, and explain the conflict between individual incentives and collective welfare.

(e) One flatmate's parent, wanting to incentivise good behaviour, rewards every contribution made in the flat — by either flatmate — lowering the effective cost from c to $c - s$ (with $0 \le s \le c$). What is the smallest reward s that makes *Contribute* a dominant strategy for each player?

<u>**Solutions**</u>

Solution to Question 1. A finite extensive form (extensive-form game-frame) with perfect information consists of:

- a finite rooted directed tree;
- a set of players $I = \{1, \dots, n\}$ and a function assigning one player to every decision node;
- a set of actions A and a function assigning one action to every directed edge, with no two edges out of the same node assigned the same action;
- a set of outcomes O and a function assigning one outcome to every terminal node.

Perfect information means that every player, when called to move, knows exactly which node has been reached, i.e. every information set is a *singleton*: there are no non-trivial information sets, equivalently no simultaneous moves and no hidden past choices.

What to add to obtain an extensive-form game. Preferences over outcomes.

Solution to Question 2.

(a) **False.** This is Matching Pennies. Checking best responses, no pure profile is a mutual best response: against H the row player prefers H but then the column player prefers T; against T the row player prefers T but then the column player prefers H, and so on cyclically. Hence there is *no* Nash equilibrium in pure strategies.

(b) **False.** In *Selten's horse* the whole game is the *only* subgame; there is no proper subgame. Recall that a subgame must start at a singleton node and may not cut through an information set. The only non-root singleton decision node is Player 2's node, but the subtree starting there contains only *one* of the two nodes of Player 3's information set (the node reached after c), while the other node (reached after D) lies outside that subtree. Starting a subgame at Player 2's node would therefore split Player 3's information set, which is not allowed. Player 3's two nodes are not singletons, so no subgame starts there either. Hence the whole game is the unique subgame and the statement is false.

Solution to Question 3.

(a) *Player 1.* Strategy C is strictly dominated by the mixed strategy

$$\sigma_1 = \begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

since against L, M, R this mixture gives Player 1 the payoffs $\frac{5}{2}, 2, 2$, while C gives $2, 1, 1$. Eliminate C.

Player 2. In the reduced game, strategy R is strictly dominated by L. Eliminate R.

The surviving game is the 2×2 game

	L	M
A	$(5,2)$	$(1,0)$
B	$(0,1)$	$(3,4)$

(b) Since strictly dominated strategies are never played in equilibrium, the Nash equilibria of the original game coincide with those of the reduced 2×2 game.

Pure equilibria: (A, L) and (B, M) (each is a mutual best response).

Mixed equilibrium: let Player 1 play A with probability p and Player 2 play L with probability q. Player 1 is indifferent when the payoffs to A and B coincide, and likewise for Player 2:

$$\underbrace{5q + 1(1-q)}_{A} = \underbrace{0\,q + 3(1-q)}_{B} \iff 4q + 1 = 3 - 3q \iff q = \tfrac{2}{7},$$

$$\underbrace{2p + 1(1-p)}_{L} = \underbrace{0\,p + 4(1-p)}_{M} \iff p + 1 = 4 - 4p \iff p = \tfrac{3}{5}.$$

Hence the third Nash equilibrium is

$$\sigma_1 = \begin{pmatrix} A & B & C \\ \frac{3}{5} & \frac{2}{5} & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} L & M & R \\ \frac{2}{7} & \frac{5}{7} & 0 \end{pmatrix}.$$

In total the game has three Nash equilibria: two in pure strategies and one in (properly) mixed strategies.

Solution to Question 4.

(a) At the root Ann chooses *Out* (terminal payoff $(2,5)$) or *In*. After *In* the two firms move simultaneously: this is modelled by one firm choosing its action at a node and the other choosing at an information set that does not reveal the first firm's choice. The node reached immediately after *In* is a singleton and the entire simultaneous game lies below it, so it starts a proper subgame. Hence the game has **two subgames**: the whole game and the simultaneous-move game after *In* (the unique proper subgame).

(b) The subgame is a Battle-of-the-Sexes game. *Pure equilibria:* (a_1, b_1) with payoff $(4,1)$ and (a_2, b_2) with payoff $(1,4)$. *Mixed equilibrium:* let Ann play a_1 with probability p and Bob play b_1 with probability q. Indifference gives

$$4q = 1 - q \iff q = \tfrac{1}{5}, \qquad p = 4(1-p) \iff p = \tfrac{4}{5}.$$

So the mixed NE is $\big(a_1 : \tfrac{4}{5},\, a_2 : \tfrac{1}{5}\big)$, $\big(b_1 : \tfrac{1}{5},\, b_2 : \tfrac{4}{5}\big)$, which gives each player expected payoff $\tfrac{4}{5}$.

(c) Replace the subgame by each of its equilibrium payoffs and let Ann choose optimally at the root, comparing with her *Out* payoff of 2:

- Continuation (a_1, b_1): Ann's continuation payoff $4 > 2$, so she plays *In*. SPE $(In;\, a_1, b_1)$, outcome $(4,1)$.
- Continuation (a_2, b_2): Ann's continuation payoff $1 < 2$, so she plays *Out*. SPE $(Out;\, a_2, b_2)$, outcome $(2,5)$.
- Mixed continuation: Ann's continuation payoff $\tfrac{4}{5} < 2$, so she plays *Out*. SPE $(Out;$ mixed$)$, outcome $(2,5)$.

There are thus three subgame-perfect equilibria; entry occurs only when the subgame is expected to coordinate on Ann's preferred standard (a_1, b_1).

(d) Consider $(Out;\, a_1, b_2)$: Ann stays *Out*, with the off-path plan a_1 for Ann and b_2 for Bob. This is a Nash equilibrium of the whole game: since *Out* is played the subgame is never reached, so Bob is indifferent across his subgame actions, and if Ann deviated to *In* the outcome (a_1, b_2) would give her $0 < 2$, so *Out* is optimal. Yet (a_1, b_2) is *not* a Nash equilibrium of the subgame (given a_1, Bob strictly prefers b_1; given b_2, Ann strictly prefers a_2). Hence the profile is not subgame-perfect: it rests on non-credible play in the subgame that is never tested on the equilibrium path.

Solution to Question 5. (For brevity write C for *Contribute* ($g_i = 1$) and N for *Not* ($g_i = 0$).)

(a) The benefit of a single contribution is b for each flatmate, while its cost c falls entirely on the contributor. So, $c > b$ means a contribution does *not* pay for the contributor alone (the private benefit b is below the cost c), so individually nobody wants to contribute. Instead, $c < 2b$ means a contribution does pay *socially*: it generates total benefit $2b$ (one b for each flatmate), which exceeds its cost c. So the assumption captures exactly the tension of a public good — privately too costly, but socially worthwhile.

(b) Using $\pi_i = b(g_1 + g_2) - c\,g_i$:

	C	N
C	$(2b - c,\ 2b - c)$	$(b - c,\ b)$
N	$(b,\ b - c)$	$(0,\ 0)$

(c) For either player, switching from N to C changes own payoff by $b - c < 0$ regardless of the other's choice:

$$(2b - c) - b = b - c < 0 \quad \text{(if the other contributes)},$$
$$(b - c) - 0 = b - c < 0 \quad \text{(if the other does not)}.$$

Hence N (free-riding) is a *strictly dominant strategy* for both players, and the unique Nash equilibrium is (N, N), with payoffs $(0, 0)$.

(d) **No.** The equilibrium (N, N) is Pareto dominated by (C, C), which gives each player $2b - c > 0$. Both players strictly prefer mutual contribution to the equilibrium, yet each individually prefers to free-ride because a single contribution returns only b to the contributor while costing $c > b$. The positive externality b that a contribution confers on the *other* flatmate is not internalized: this is the free-rider problem, and it is exactly a Prisoner's-Dilemma structure.

(e) With effective cost $c - s$, the gain from switching to C becomes $b - (c - s) = b - c + s$. *Contribute* is (weakly) dominant once this is nonnegative, i.e. $s \geq c - b$, and strictly dominant for $s > c - b$. The smallest such reward is

$$s^* = c - b.$$

A reward of $s^* = c - b$ brings the private cost down to the private benefit, so that the parent's incentive exactly offsets the temptation to free-ride. Since the reward is paid for *any* contribution in the flat, it lowers the cost symmetrically for both flatmates: each now finds it (weakly) dominant to contribute, and the efficient outcome (C, C) is restored.