Game Theory — Exam - A

Enrico Mattia Salonia

2 April 2026

Time: 75 minutes. Be clear and concise.

Part I — Definitions and True/False

Question 1 (4 points)**.** Define a Nash Equilibrium strategy profile in an ordinal game in strategic form with n players.

Question 2 (7 points)**.** Say whether the following statements are true, false, or uncertain, and *justify* your answer. ("uncertain" means that the statement is true under conditions that are not explicitly mentioned.)

(a) [**3.5 points**] Consider the following game: Ann has an apple, while Bob has a banana. They simultaneously choose whether to give their fruit to the other or to keep it. **Statement:** "Each player in this game has a strictly dominant strategy."

(b) [**3.5 points**] **Statement:** "In the following extensive-form game with imperfect information, each player has two pure strategies."

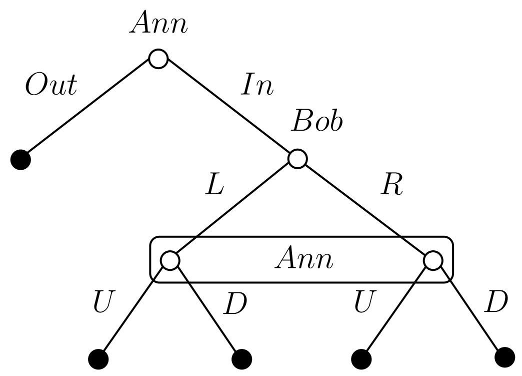

Figure 1: Extensive-form game for statement (b).

Part II — Exercises

Question 3 (7 points)**.** Consider the following strategic-form game with two players who have expected utility (von Neumann-Morgenstern) preferences:

	L	M	R
A	$(8, 1)$	$(2, 7)$	$(5, 0)$
B	$(1, 8)$	$(7, 2)$	$(5, 0)$
C	$(1, 3)$	$(2, 3)$	$(0, 9)$

(a) Perform the Cardinal Iterated Deletion of Strictly Dominated Strategies.

(b) Find all Nash equilibria in mixed strategies of the game.

Question 4 (7 points)**.** Firm 2 is an incumbent monopolist earning a profit of 2 (million euros). Firm 1 is a potential entrant that currently earns a profit of 2 in another market. Firm 1 moves first and decides whether to *Enter* this market or *Stay Out*.

- If Firm 1 stays out, both firms keep their current profits.
- If Firm 1 enters, Firm 2 observes the entry and chooses whether to *Fight* (start a price war) or *Accommodate* (share the market).

– If Firm 2 fights, the price war drives Firm 1's profit to 0 and Firm 2's profit to 1.
– If Firm 2 accommodates, Firm 1 earns 4 and Firm 2 earns 3.

Assume that for both firms payoffs equal profits.

(a) Draw the tree of the extensive-form game with perfect information.

(b) Write the strategic form of the game and find all pure-strategy Nash equilibria.

(c) Solve the game by backward induction and find all the subgame-perfect equilibria.

(d) Are there Nash equilibria that are not subgame-perfect? If yes, why?

Part III — Advanced Question

Question 5 (6 points)**.** Three students must decide whether to collectively hire a private tutor for a game theory exam review session. The cost of the tutor would be split equally, so $c_1 = c_2 = c_3 = 15$. For every student $i = 1, 2, 3$, let v_i be the gross benefit from having the tutor hired. The gross benefits are $v_1 = 25$, $v_2 = 20$, and $v_3 = 5$. The net benefit to student i is $v_i - c_i$. Student i has the following utility of wealth function (where m_i denotes student i's wealth):

$$U_i(\$m_i) = \begin{cases} m_i & \text{if the tutor is not hired,} \\ m_i + v_i & \text{if the tutor is hired.} \end{cases}$$

Assume all students are rich enough to afford any cost.

The pivotal mechanism. The students use the following procedure to decide whether to hire the tutor.

Step 1. Each student i simultaneously announces a number w_i, interpreted as their gross benefit from having the tutor hired.

Step 2. Compute $D = \sum_{i=1}^{3}(w_i - c_i)$. If $D > 0$ the tutor is hired; if $D \leq 0$ the tutor is not hired.

Step 3. Pivotality. Student i is said to be *pivotal* if removing her announcement would reverse the decision. Formally, let $D_{-i} = \sum_{j \neq i}(w_j - c_j)$. Student i is pivotal if the sign of D_{-i} is opposite to the sign of D (or if $D_{-i} = 0$ and $D \neq 0$).

Step 4. Tax. A pivotal student i pays a tax equal to $|D_{-i}|$; a non-pivotal student pays no tax. The tax is paid on top of the cost share c_i if the project is carried out.

(a) What is the Pareto-efficient decision: to hire the tutor or not?

(b) As you know, in the pivotal mechanism each student has a weakly dominant strategy. If all the students played their weakly dominant strategies, would the tutor be hired? Who is pivotal and what tax does each student pay?

(c) Assume now that the students try to fund the tutor through *voluntary contributions*. Each student i simultaneously chooses a contribution $c_i \in [0, e]$, where e is the student's budget. The amount collected is used to improve the quality of the tutoring session, benefiting everyone. Payoffs are

$$\pi_i(c_i, c_{-i}) = e - c_i + r\sum_{j=1}^{3} c_j, \qquad \tfrac{1}{3} < r < 1.$$

Explain briefly: What is each student's dominant strategy in this voluntary contribution game? Is the resulting equilibrium Pareto efficient? Can you now give a rationale for using the pivotal mechanism studied in parts (a)–(b)?

Solutions

Solution to Question 1. Given an ordinal game in strategic form with n players,

$$\langle I, (S_1, \dots, S_n), O, f, (\succsim_1, \dots, \succsim_n)\rangle,$$

a strategy profile $s^* = (s_1^*, \dots, s_n^*) \in S_1 \times \cdots \times S_n$ is a Nash equilibrium if, for every player $i = 1, \dots, n$,

$$\pi_i(s^*) \geq \pi_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*) \quad \text{for all } s_i \in S_i.$$

Equivalently, each player's strategy in s^* is a best response to the strategies of the other players; no player can gain from a unilateral deviation.

Solution to Question 2.

(a) Uncertain. The statement "Each player has a strictly dominant strategy" depends on preferences, which are not specified. If each player only cares about receiving the other fruit (and not about giving away their own fruit), then "Give" is strictly dominant for both players. But if a player dislikes giving away their fruit (or values keeping it enough), "Give" need not dominate "Keep." Therefore, without explicit payoff numbers or preference assumptions, the statement cannot be determined.

(b) False. Bob has one decision node (reached after In), so Bob has exactly two pure strategies: L and R.

Ann instead has two information sets (the initial node and the information set grouping the nodes reached after L and R), so a pure strategy must specify one action at each of these information sets. Therefore Ann has four pure strategies: $(In, U), (In, D), (Out, U), (Out, D)$.

Solution to Question 3.

(a) For Player 1, strategy C is strictly dominated by the mixed strategy

$$\sigma_1 = \begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

since against L, M, R, this mixture gives payoffs $4.5, 4.5, 5$, while C gives $1, 2, 0$. So eliminate C.

In the reduced game $\{A, B\} \times \{L, M, R\}$, strategy R for Player 2 is strictly dominated by the pure strategy L, because against A, L gives payoff $1 > 0$, and against B, L gives payoff $8 > 0$. So eliminate R.

The surviving game is

	L	M
A	$(8, 1)$	$(2, 7)$
B	$(1, 8)$	$(7, 2)$

(b) No pure equilibrium exists in the reduced 2×2 game.

Let Player 1 play A with probability p, and Player 2 play L with probability q. Indifference conditions are:

$$8q + 2(1-q) = 1q + 7(1-q) \iff q = \tfrac{5}{12},$$
$$1p + 8(1-p) = 7p + 2(1-p) \iff p = \tfrac{1}{2}.$$

Therefore the mixed-strategy Nash equilibrium is

$$\sigma_1 = \begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} L & M & R \\ \frac{5}{12} & \frac{7}{12} & 0 \end{pmatrix}.$$

Solution to Question 4.

(a) The tree has Firm 1 at the root choosing *Enter* or *Stay Out*. The branch *Stay Out* ends with payoff $(2, 2)$. If *Enter*, Firm 2 chooses *Fight* (payoff $(0, 1)$) or *Accommodate* (payoff $(4, 3)$).

(b) Strategic form:

	Fight	*Accommodate*
Enter	$(0, 1)$	$(4, 3)$
Stay Out	$(2, 2)$	$(2, 2)$

Best responses imply two pure-strategy Nash equilibria:

$$(\textit{Stay Out}, \textit{Fight}), \qquad (\textit{Enter}, \textit{Accommodate}).$$

(c) Backward induction: at Firm 2's node, *Accommodate* is optimal since $3 > 1$. Anticipating this, Firm 1 chooses *Enter* since $4 > 2$. Hence the unique subgame-perfect equilibrium is

$$(\textit{Enter}, \textit{Accommodate}).$$

(d) The equilibrium (*Stay Out*, *Fight*) relies on a non-credible threat: "if Firm 1 enters, Firm 2 would fight." This is not credible because in the subgame after *Enter*, Firm 2 strictly prefers *Accommodate* to *Fight* $(3 > 1)$.

Solution to Question 5.

(a) We check whether the sum of net benefits is positive:

$$\sum_{i=1}^{3}(v_i - c_i) = (25 - 15) + (20 - 15) + (5 - 15) = 10 + 5 - 10 = 5 > 0.$$

Since the total net benefit is positive, it is **Pareto efficient to hire the tutor**.

(b) In the pivotal mechanism, the weakly dominant strategy is truthful reporting: $w_i^* = v_i$ for each student.

Under truthful reporting:

$$\sum_{i=1}^{3}(v_i - c_i) = 10 + 5 - 10 = 5 > 0.$$

so the tutor is hired.

Pivotality under truthful play (hiring case):

- Without Student 1: $5 + (-10) = -5 < 0 \;\Rightarrow\;$ **Pivotal**, tax $= 5$.
- Without Student 2: $10 + (-10) = 0$, and $D = 5 \neq 0 \;\Rightarrow\;$ **Pivotal**, tax $= |0| = 0$.
- Without Student 3: $10 + 5 = 15 > 0 \;\Rightarrow\;$ Not pivotal, tax $= 0$.

Student	1	2	3
Pivotal?	Yes	Yes	No
Tax	5	0	0

(c) Student i's payoff in the voluntary contribution game is $\pi_i = e + (r - 1)c_i + r\sum_{j\neq i} c_j$. Since $r < 1$, payoff is decreasing in own c_i, so each student's dominant strategy is $c_i = 0$.

Hence the unique equilibrium is $(0, 0, 0)$, which is not Pareto efficient, because aggregate welfare increases with total contributions when $3r - 1 > 0$ (here $r > \frac{1}{3}$).

This free-rider problem motivates mechanisms such as the pivotal mechanism, which align private incentives with efficient provision of the public good.

Game Theory — Exam - B

Enrico Mattia Salonia

2 April 2026

Time: 75 minutes. Be clear and concise.

Part I — Definitions and True/False

Question 1 (4 points)**.** Define a Nash Equilibrium strategy profile in an ordinal game in strategic form with n players.

Question 2 (7 points)**.** Say whether the following statements are true, false, or uncertain, and *justify* your answer. ("uncertain" means that the statement is true under conditions that are not explicitly mentioned.)

(a) [**3.5 points**] Consider the following game: Ann has an apple, while Bob has a banana. They simultaneously choose whether to give their fruit to the other or to keep it. **Statement:** "Each player in this game has a strictly dominant strategy."

(b) [**3.5 points**] **Statement:** "In the following extensive-form game with imperfect information, each player has two pure strategies."

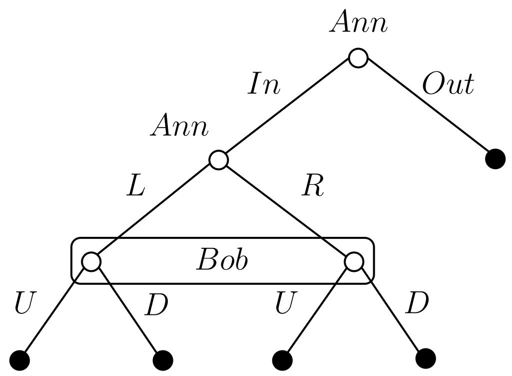

Figure 1: Extensive-form game for statement (b).

Part II — Exercises

Question 3 (7 points)**.** Consider the following strategic-form game with two players who have expected utility (von Neumann-Morgenstern) preferences:

	L	M	R
A	$(9,1)$	$(3,7)$	$(6,0)$
B	$(1,9)$	$(7,3)$	$(6,0)$
C	$(1,4)$	$(3,4)$	$(0,8)$

(a) Perform the Cardinal Iterated Deletion of Strictly Dominated Strategies.

(b) Find all Nash equilibria in mixed strategies of the game.

Question 4 (7 points)**.** Firm 2 is an incumbent monopolist earning a profit of 3 (million euros). Firm 1 is a potential entrant that currently earns a profit of 3 in another market. Firm 1 moves first and decides whether to *Enter* this market or *Stay Out*.

- If Firm 1 stays out, both firms keep their current profits.
- If Firm 1 enters, Firm 2 observes the entry and chooses whether to *Fight* (start a price war) or *Accommodate* (share the market).

– If Firm 2 fights, the price war drives Firm 1's profit to 1 and Firm 2's profit to 0.

– If Firm 2 accommodates, Firm 1 earns 5 and Firm 2 earns 4.

Assume that for both firms payoffs equal profits.

(a) Draw the tree of the extensive-form game with perfect information.

(b) Write the strategic form of the game and find all pure-strategy Nash equilibria.

(c) Solve the game by backward induction and find all the subgame-perfect equilibria.

(d) Are there Nash equilibria that are not subgame-perfect? If yes, why?

Part III — Advanced Question

Question 5 (6 points)**.** Three students must decide whether to collectively hire a private tutor for a game theory exam review session. The cost of the tutor would be split equally, so $c_1 = c_2 = c_3 = 12$. For every student $i = 1, 2, 3$, let v_i be the gross benefit from having the tutor hired. The gross benefits are $v_1 = 25$, $v_2 = 18$, and $v_3 = 4$. The net benefit to student i is $v_i - c_i$. Student i has the following utility of wealth function (where m_i denotes student i's wealth):

$$U_i(\$m_i) = \begin{cases} m_i & \text{if the tutor is not hired,} \\ m_i + v_i & \text{if the tutor is hired.} \end{cases}$$

Assume all students are rich enough to afford any cost.

The pivotal mechanism. The students use the following procedure to decide whether to hire the tutor.

Step 1. Each student i simultaneously announces a number w_i, interpreted as their gross benefit from having the tutor hired.

Step 2. Compute $D = \sum_{i=1}^{3}(w_i - c_i)$. If $D > 0$ the tutor is hired; if $D \leq 0$ the tutor is not hired.

Step 3. Pivotality. Student i is said to be *pivotal* if removing her announcement would reverse the decision. Formally, let $D_{-i} = \sum_{j \neq i}(w_j - c_j)$. Student i is pivotal if the sign of D_{-i} is opposite to the sign of D (or if $D_{-i} = 0$ and $D \neq 0$).

Step 4. Tax. A pivotal student i pays a tax equal to $|D_{-i}|$; a non-pivotal student pays no tax. The tax is paid on top of the cost share c_i if the project is carried out.

(a) What is the Pareto-efficient decision: to hire the tutor or not?

(b) As you know, in the pivotal mechanism each student has a weakly dominant strategy. If all the students played their weakly dominant strategies, would the tutor be hired? Who is pivotal and what tax does each student pay?

(c) Assume now that the students try to fund the tutor through *voluntary contributions*. Each student i simultaneously chooses a contribution $c_i \in [0, e]$, where e is the student's budget. The amount collected is used to improve the quality of the tutoring session, benefiting everyone. Payoffs are

$$\pi_i(c_i, c_{-i}) = e - c_i + r\sum_{j=1}^{3} c_j, \qquad \tfrac{1}{3} < r < 1.$$

Explain briefly: What is each student's dominant strategy in this voluntary contribution game? Is the resulting equilibrium Pareto efficient? Can you now give a rationale for using the pivotal mechanism studied in parts (a)–(b)?

Solutions

Solution to Question 1. Given an ordinal game in strategic form with n players,

$$\langle I, (S_1, \ldots, S_n), O, f, (\succsim_1, \ldots, \succsim_n)\rangle,$$

a strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S_1 \times \cdots \times S_n$ is a Nash equilibrium if, for every player $i = 1, \ldots, n$,

$$\pi_i(s^*) \geq \pi_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \quad \text{for all } s_i \in S_i.$$

Equivalently, each player's strategy in s^* is a best response to the strategies of the other players; no player can gain from a unilateral deviation.

Solution to Question 2.

(a) **Uncertain.** The statement "Each player has a strictly dominant strategy" depends on preferences, which are not specified. If each player only cares about receiving the other fruit (and not about giving away their own fruit), then "Give" is strictly dominant for both players. But if a player dislikes giving away their fruit (or values keeping it enough), "Give" need not dominate "Keep." Therefore, without explicit payoff numbers or preference assumptions, the statement cannot be determined.

(b) **False.** Bob has one information set with two available actions, so Bob has exactly two pure strategies: U and D.

Ann instead has two decision nodes (the initial node and the node reached after In), so a pure strategy must specify one action at each of these nodes. Therefore Ann has four pure strategies: $(In, L), (In, R), (Out, L), (Out, R)$.

Solution to Question 3.

(a) For Player 1, strategy C is strictly dominated by the mixed strategy

$$\sigma_1 = \begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

since against L, M, R, this mixture gives payoffs $5, 5, 6$, while C gives $1, 3, 0$. So eliminate C.

In the reduced game $\{A, B\} \times \{L, M, R\}$, strategy R for Player 2 is strictly dominated by the pure strategy L, because against A, L gives payoff $1 > 0$, and against B, L gives payoff $9 > 0$. So eliminate R.

The surviving game is

	L	M
A	$(9, 1)$	$(3, 7)$
B	$(1, 9)$	$(7, 3)$

(b) No pure equilibrium exists in the reduced 2×2 game.

Let Player 1 play A with probability p, and Player 2 play L with probability q. Indifference conditions are:

$$9q + 3(1 - q) = 1q + 7(1 - q) \iff q = \tfrac{1}{3},$$
$$1p + 9(1 - p) = 7p + 3(1 - p) \iff p = \tfrac{1}{2}.$$

Therefore the mixed-strategy Nash equilibrium is

$$\sigma_1 = \begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} L & M & R \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}.$$

Solution to Question 4.

(a) The tree has Firm 1 at the root choosing *Enter* or *Stay Out*. The branch *Stay Out* ends with payoff $(3, 3)$. If *Enter*, Firm 2 chooses *Fight* (payoff $(1, 0)$) or *Accommodate* (payoff $(5, 4)$).

(b) Strategic form:

	Fight	*Accommodate*
Enter	$(1, 0)$	$(5, 4)$
Stay Out	$(3, 3)$	$(3, 3)$

Best responses imply two pure-strategy Nash equilibria:

$$(\textit{Stay Out}, \textit{Fight}), \qquad (\textit{Enter}, \textit{Accommodate}).$$

(c) Backward induction: at Firm 2's node, *Accommodate* is optimal since $4 > 0$. Anticipating this, Firm 1 chooses *Enter* since $5 > 3$. Hence the unique subgame-perfect equilibrium is

$$(\textit{Enter}, \textit{Accommodate}).$$

(d) The equilibrium (*Stay Out*, *Fight*) relies on a non-credible threat: "if Firm 1 enters, Firm 2 would fight." This is not credible because in the subgame after *Enter*, Firm 2 strictly prefers *Accommodate* to *Fight* $(4 > 0)$.

Solution to Question 5.

(a) We check whether the sum of net benefits is positive:

$$\sum_{i=1}^{3}(v_i - c_i) = (25 - 12) + (18 - 12) + (4 - 12) = 13 + 6 - 8 = 11 > 0.$$

Since the total net benefit is positive, it is **Pareto efficient to hire the tutor**.

(b) In the pivotal mechanism, the weakly dominant strategy is truthful reporting: $w_i^* = v_i$ for each student.

Under truthful reporting:

$$\sum_{i=1}^{3}(v_i - c_i) = 13 + 6 - 8 = 11 > 0.$$

so the tutor is hired.

Pivotality under truthful play (hiring case):

- Without Student 1: $6 + (-8) = -2 < 0 \Rightarrow$ **Pivotal**, tax $= 2$.
- Without Student 2: $13 + (-8) = 5 > 0 \Rightarrow$ Not pivotal, tax $= 0$.
- Without Student 3: $13 + 6 = 19 > 0 \Rightarrow$ Not pivotal, tax $= 0$.

Student	1	2	3
Pivotal?	Yes	No	No
Tax	2	0	0

(c) Student i's payoff in the voluntary contribution game is $\pi_i = e + (r-1)c_i + r\sum_{j \neq i} c_j$. Since $r < 1$, payoff is decreasing in own c_i, so each student's dominant strategy is $c_i = 0$.

Hence the unique equilibrium is $(0, 0, 0)$, which is not Pareto efficient, because aggregate welfare increases with total contributions when $3r - 1 > 0$ (here $r > \frac{1}{3}$).

This free-rider problem motivates mechanisms such as the pivotal mechanism, which align private incentives with efficient provision of the public good.