Microeconomics I Lecture Notes

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This is a first draft of notes that I plan to improve and expand.
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Contents

Preamble

These notes accompany the first part of the PhD microeconomics sequence. They cover **choice under uncertainty** and **general equilibrium theory**. The write-up is still a work in progress, and I will continue to update it. If you spot any mistakes or typos, please let me know.

I have aimed for a conversational tone rather than the more formal and encyclopedic style of, say, Mas-Colell et al. (1995). I assume no prior knowledge of the topics, though—as usual—some mathematical maturity helps (and I hope you will develop it along the way!). Each lecture summarizes what we cover in class, followed by exercises and suggestions for further reading. Whenever a result is proved, I have tried to give the simplest proof available. This often makes explanations and proofs a bit longer than strictly necessary, but, I hope, also more accessible.

Before diving in, you might enjoy some non-technical background that helps frame the topics we will study: Kreps (1988, ch. 1), Debreu (1959, pp. ix–xi), Myerson (1997, pp. 1–7), and Gilboa (2009, chs. 1–2).

You will occasionally see smaller text like this. These remarks are not essential for following the main exposition, but they add context or point to related ideas. Feel free to skip them on a first pass.

These notes draw on several sources. The main reference is Mas-Colell et al. (1995), but both here and in the text you will find pointers to alternative or complementary readings. A short reading list follows. If you would like more references or wish to discuss any of the material, just send me an email—I am always happy to talk.

Have fun!

Choice under uncertainty.

- **Mas-Colell et al. (1995), ch. 6.**
- Kreps (1988), chs. 4-6.
- Fishburn (1970), ch. 8.
- Kreps (2013), chs. 5-6.
- Gilboa (2009).

General equilibrium theory.

- **Mas-Colell et al. (1995) chs. 15–17.**
- Thomson (2011), sec 4.3.
- Kreps (2013), chs. 14–15.
- Debreu (1959).
- Hildenbrand & Kirman (1976).

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Lecture 1

Introduction to uncertainty

"We are who we are. Lotteries are stupid."

House (2011)

1.1 How to model uncertainty

The outcomes of our decisions are often uncertain, so we need a choice theory that takes uncertainty into account. Let us begin by thinking about how to represent uncertainty. Suppose you make a bet with a friend: if a fair coin toss results in heads, you receive 10 euros; otherwise, you pay 10 euros to your friend. There are two possible outcomes, 10 and -10, and since the coin is fair, each occurs with probability $1/2$. What are the main ingredients of this example?

First, we started from a set of possible outcomes—in this case, the monetary transfers 10 and -10. Second, we specified the probability of each outcome occurring, $1/2$ for both. We call such an object—a set of outcomes, each associated with a probability—a **lottery**. Denote the set of outcomes by X. Generic elements of X will be written x, y, z, or sometimes $x_1, x_2, \ldots$. For simplicity, assume that X is finite. Outcomes alone are not enough to describe a lottery: we also need a probability distribution over outcomes, as in the $1/2$–$1/2$ distribution of the fair coin above. The set of all lotteries over X is denoted by $\Delta(X)$.[1] Each element of $\Delta(X)$ is a function $p\colon X \to [0,1]$ such that $\sum_{x \in X} p(x) = 1$; it maps each outcome x to a number $p(x) \in [0,1]$, representing the probability that x occurs.[2] We can equivalently represent a lottery as a vector, for example $p = (p(x), p(y), p(z))$ if $X = \{x, y, z\}$.

Example 1.1. In the example above, the set of outcomes is $\{10, -10\}$, and the lottery $p \in \Delta(\{10, -10\})$ induced by the fair coin toss satisfies $p(10) = p(-10) = 1/2$. ■

We can depict lotteries using a tree diagram, as in Figure 1.1.

[1]Why the notation Δ? You will see soon.
[2]Why do we write a sum $\sum_{x \in X} p(x) = 1$ rather than an integral?

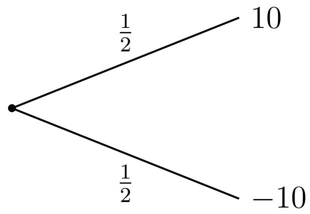


Figure 1.1: Lottery from Example 1.1.

Remark 1.1. *Notice that in this setup we are missing something: whether the coin lands on heads or tails is irrelevant; only the probabilities of the outcomes matter, not the events that generate them. This is a limitation of the model, which we will address later when we introduce a state-space representation of uncertainty.*

The set of lotteries $\Delta(X)$ has *structure*: we can combine its elements in a meaningful way. For example, consider a lottery r that yields lottery p with probability α and lottery q with probability $1-\alpha$, where $\alpha \in [0,1]$. Such an object is called a **compound lottery**. It is still an element of $\Delta(X)$, and we write $r = \alpha p + (1-\alpha)q$.

For instance, if $p(10) = 1/2$ and $q(10) = 1/4$, the associated compound lottery is shown on the left of Figure 1.2. We can compute the probability that outcome 10 occurs in this compound lottery:

$$\alpha \times 1/2 + (1-\alpha) \times 1/4 = \tfrac{1+\alpha}{4}.$$

By calculating the probability of each outcome in a compound lottery, we can *reduce* it to an equivalent simple lottery, as shown on the right of Figure 1.2.

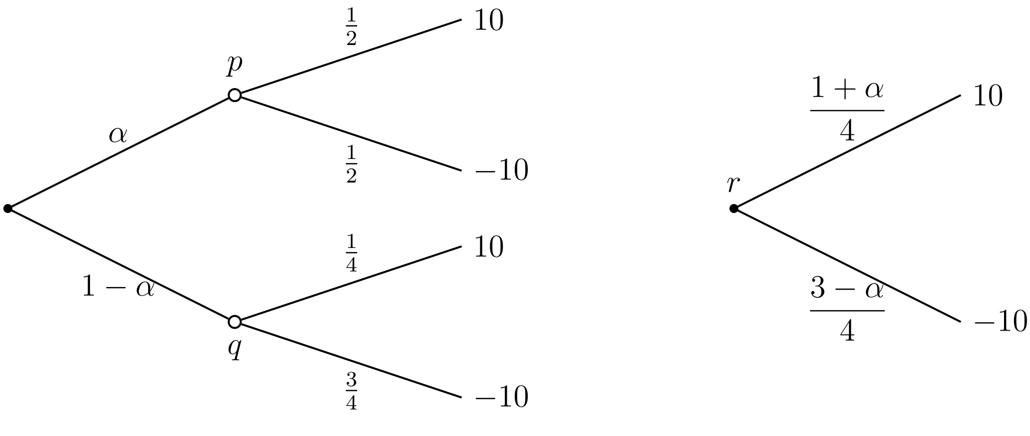


Figure 1.2: Compound lottery (left) and its reduced form (right).

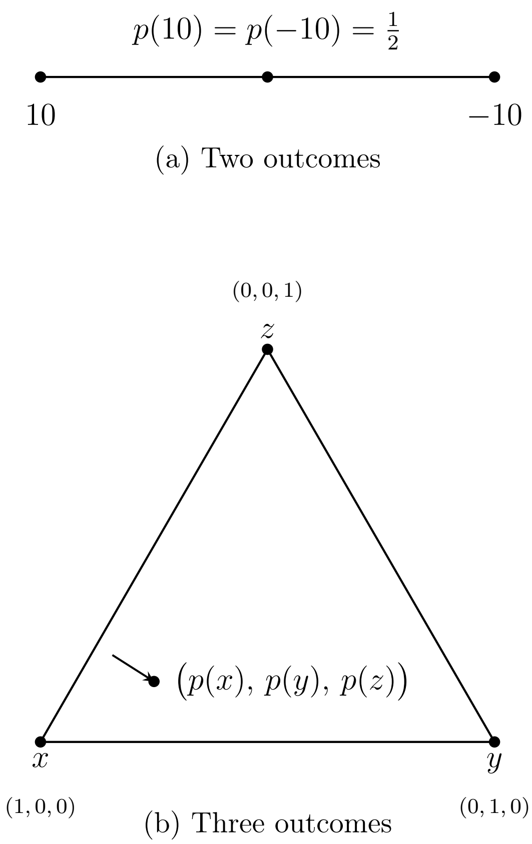
We assume *reduction of compound lotteries*: individuals are indifferent between any compound lottery and its reduced form—that is, any two lotteries that induce the same probabilities over outcomes are treated as equivalent.

Can you think of reasons why someone might *not* be indifferent between a compound lottery and its reduced form? Violations of reduction generate interesting phenomena studied in behavioural economics. See, for example, Segal (1990) and Dillenberger & Raymond (2020).

This lottery *mixing* operation would not make sense with an unstructured set of outcomes. As an illustration, suppose the set of outcomes consists of fruits. We can have an apple or a banana, but there is no fruit that is a mixture of an apple and a banana. Imposing structure on the set of elements to be ranked is one of the key moves in microeconomic theory. In fact, we will later assume that the set of outcomes is $\mathbb{R}$, the set of real numbers representing monetary outcomes, which allows us to say more than we could with a generic set of outcomes.

There is another useful way to represent lotteries graphically. Consider again the coin toss that yields 10 euros with probability $1/2$ and -10 euros with probability $1/2$. We can represent this lottery as the midpoint of the line segment whose endpoints correspond to the degenerate lotteries that yield 10 and -10 with probability 1; see panel (a) of Figure 1.3. More generally, with n possible outcomes we can represent a lottery as a point in an $(n-1)$-dimensional simplex. For example, with three outcomes we can represent lotteries as points in an equilateral triangle, as in panel (b).[3] The vertices of the triangle correspond to degenerate lotteries that yield one outcome with probability 1, while any other point in the triangle represents a lottery that yields each of the three outcomes with some probability. Roughly speaking, the farther a point is from a vertex, the lower the probability of the corresponding outcome. For example, the lottery p in panel (b) yields outcome x with relatively high probability and outcomes y and z with relatively low probabilities.

[3] That's why the Δ notation!



(a) Two outcomes

(b) Three outcomes

Figure 1.3: Lotteries as points in simplexes: (a) a two–outcome lottery lies on a line segment; (b) with three outcomes, lotteries lie in an equilateral triangle.

For a finite outcome set X, the probability simplex over X is

$$\Delta(X) = \left\{ p\colon X \to [0,1] \;\middle|\; \sum_{x \in X} p(x) = 1 \right\},$$

or equivalently,

$$\left\{ \big(p(x_1), \ldots, p(x_n)\big) \in \mathbb{R}^n \;\middle|\; p(x_i) \geq 0,\ \sum_i p(x_i) = 1 \right\}.$$

This set is an $(n-1)$-dimensional simplex whose vertices correspond to the degenerate lotteries (unit vectors), e.g. $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$.

1.2 Preferences over lotteries

Our goal is to understand how individuals choose between lotteries, whether they like or dislike risk, and how we can compare different individuals' attitudes toward risk. To do so, we need a way to express statements such as "an individual weakly prefers lottery p to lottery q". Introduce a binary relation $\succsim$ over $\Delta(X)$, where $p \succsim q$ reads "the individual weakly prefers lottery p to lottery q".[4] Contrary to choice under certainty, we are now comparing lotteries—that is, probability distributions over outcomes—rather than outcomes themselves.

Technically, $\succsim$ is a subset of $\Delta(X) \times \Delta(X)$: a set of ordered pairs of lotteries. For example, if $p, q \in \Delta(X)$, the statement "p is (weakly) preferred to q" is equivalent to $(p, q) \in \succsim$.

Recall that we can define strict preference and indifference in terms of weak preference. We write $p \succ q$, which reads "p is strictly preferred to q", if and only if $p \succsim q$ but not $q \succsim p$; and we write $p \sim q$, which reads "p is indifferent to q", if and only if both $p \succsim q$ and $q \succsim p$.

In principle, we could describe the preference relation $\succsim$ by listing, pair by pair, which lotteries are weakly preferred to which others. However, that would be rather inconvenient as a way to work with preferences. It is more practical to have a function that assigns a number to each lottery, so that we can compare lotteries by comparing their associated numbers. Such a function should "agree" with the preference relation $\succsim$ in the sense that, if p is weakly preferred to q, then the number assigned to p should be at least as large as the number assigned to q. This leads us to the notion of a **utility function** representing preferences.

Definition 1.1. *A utility function $U \colon \Delta(X) \to \mathbb{R}$* ***represents*** *the preference relation $\succsim$ over $\Delta(X)$ if, for all lotteries p, q,*

$$p \succsim q \iff U(p) \geq U(q).$$

What would be a reasonable utility function representing preferences over lotteries? A natural candidate is the **expected utility** function, defined as follows.

Definition 1.2. *Preferences $\succsim$ are represented by an* ***expected utility function*** *if there exists a function $u \colon X \to \mathbb{R}$ such that, for all lotteries p,*

$$U(p) = \sum_x p(x)\, u(x). \tag{1.1}$$

In other words, an expected utility function assigns to each lottery p the *expected value* of the function u over the outcomes, where the expectation is taken with respect to the probability distribution p. The function u is sometimes called the **Bernoulli utility function**. An expected utility function is *linear in probabilities*; that is, for any lotteries p, q and any $\alpha \in [0, 1]$,

$$U\big(\alpha p + (1-\alpha) q\big) = \alpha U(p) + (1-\alpha) U(q),$$

[4] Are you curious why we use the symbol $\succsim$ for preferences instead of $\geq$? The historian of economic theory Ivan Boldyrev told me that it originates from Herstein & Milnor (1953), who used it in their classic paper providing an axiomatic characterization of expected utility—which we will encounter soon.

meaning that the expected utility of a mixture of lotteries is the weighted average of their expected utilities. The other direction of the statement is also true: if U is linear, then it is an expected utility function.[5] Linearity is an extremely convenient property in applications, which partially explains the success of expected utility theory.

Suppose you observe an individual's choices and want to test whether their preferences can be represented by an expected utility function. How might you do that? One idea would be to design a choice task, predict the individual's choices using Equation (1.1), and see whether the predictions are accurate. However, this approach is hard to apply, because to make predictions you would need to assume a specific function u. This is sometimes done—certain functional forms work particularly well—but there is another approach.

We can instead look for *behavioural predictions* that are independent of any specific u; that is, properties of choices that any expected utility maximiser must satisfy. If we can identify such properties, we can design a choice task aimed at testing whether the individual's choices satisfy them. Linearity of expected utility is one such property: it holds regardless of the specific u, and it is the main behavioural prediction of expected utility theory.

In the next lecture, we will examine properties that *fully characterise* preferences representable by an expected utility function. In other words, violating these properties implies that preferences cannot be represented by an expected utility function, while satisfying them implies that they can be represented *only* by an expected utility function. Such characterisations are remarkably powerful. We will also discuss a second point of view on the role of these properties in defining a theory of choice.

Things to read. See Kreps (1988, pp. 31–33) for a brief, intuitive introduction to the lottery model in this chapter. For a similar treatment in a standard textbook, see Mas-Colell et al. (1995, pp. 168–170).

1.3 Exercises

Exercise 1.1. Can we still represent the set of lotteries and compound lotteries on the simplex if individuals are *not* indifferent between a compound lottery and its reduced form? Why or why not?

Solution to Exercise 1.1. If individuals are not indifferent between a compound lottery and its reduced form, then one would need to study preferences over $\Delta(\Delta(X))$, the set of lotteries over lotteries. This set is different from $\Delta(X)$ itself, so the simplex representation of lotteries would not work. A compound lottery, an element of $\Delta(\Delta(X))$, cannot be represented as a point in the simplex of $\Delta(X)$ because it is a different object. □

[5] You are asked to prove this in Exercise 1.6.

Exercise 1.2. Assume there are three outcomes x, y, z. Draw, in the simplex, the set of lotteries that yield each outcome with the same probability and the lottery that yields x with certainty. Now draw the set of all mixtures of these two lotteries. Assume that the individual is indifferent between the lottery yielding each outcome with the same probability, the lottery yielding x with certainty, and any mixture of the two. Which part of the simplex does this indifference "curve" correspond to? Is it really a curve?

Exercise 1.3. Assuming three outcomes x, y, z, draw in the simplex the set of lotteries that yield outcome x with probability at least $1/2$.

Exercise 1.4. In the main text, we assumed that individuals are indifferent between a compound lottery and its reduced form. State this indifference formally as a condition on the preference relation $\succsim$, you might need to develop the appropriate notation. If you are stuck, check Jehle & Reny (2011, ch. 4).

Solution to Exercise 1.4. Let $p, q \in \Delta(X)$ be two lotteries, and let $\alpha \in [0, 1]$. Consider the compound lottery $r = \alpha p + (1 - \alpha)q$. As discussed in Exercise 1.1, r is an element of $\Delta(\Delta(X))$. Let r' be the reduced form of r, defined by $r'(x) = \alpha p(x) + (1 - \alpha)q(x)$ for all x. One can identify r' as an element of the set of compound lotteries $\Delta(\Delta(X))$, it is a "degenerate" compound lottery assigning probability 1 to a specific lottery. Preferences $\succsim$ are then over $\Delta(\Delta(X))$. Reduction of compound lotteries states that $r \sim r'$; that is, the individual is indifferent between the compound lottery and its reduced form. □

Exercise 1.5. Show that preferences represented by an expected utility function satisfy reduction of compound lotteries. It might be useful to use what you learned from Exercise 1.4.

Solution to Exercise 1.5. Let $p, q \in \Delta(X)$ be two lotteries, and let $\alpha \in [0, 1]$. Consider the compound lottery $r = \alpha p + (1 - \alpha)q$, and let r' be its reduced form, defined by $r'(x) = \alpha p(x) + (1 - \alpha)q(x)$ for all x. We want to show that $r \sim r'$, which is equivalent to showing that $U(r) = U(r')$. By definition of expected utility,

$$U(r) = \sum_x r(x)u(x).$$

However, since r is a compound lottery, we can express $U(r)$ as follows:

$$U(r) = \alpha U(p) + (1 - \alpha)U(q).$$

Substituting the definitions of $U(p)$ and $U(q)$, we have

$$U(r) = \alpha \sum_x p(x)u(x) + (1 - \alpha) \sum_x q(x)u(x).$$

Combining the sums, we get

$$U(r) = \sum_x [\alpha p(x) + (1-\alpha) q(x)] u(x).$$

Notice that the term in brackets is exactly $r'(x)$. Therefore,

$$U(r) = \sum_x r'(x) u(x) = U(r').$$

□

Exercise 1.6. Show that a function U is an expected utility function if and only if it is linear in probabilities.

Solution to Exercise 1.6. Check Mas-Colell et al. (1995, Proposition 6.B.5, p. 173). □

Exercise 1.7. An individual faces two choice problems. In the first problem, they choose between receiving 50 euros for sure and a lottery that yields 250 euros with probability 0.10, 50 euros with probability 0.89, and 0 euros with probability 0.01. In the second problem, they choose between two lotteries: the first yields 50 euros with probability 0.11 and 0 euros with probability 0.89; the second yields 250 euros with probability 0.10 and 0 euros with probability 0.90. Suppose that in the first problem the individual prefers the sure amount to the lottery, while in the second they prefer the second lottery (yielding 250 euros with probability 0.10) to the first (yielding 50 euros with probability 0.11). Assume the individual prefers having more money. Can these preferences be represented by an expected utility function? Why or why not?

Solution to Exercise 1.7. Check Mas-Colell et al. (1995, Example 6.B.2, p. 179). □

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Lecture 2

Expected utility theory

2.1 Assumptions on preferences

We now impose properties on preferences over lotteries and study their behavioural implications. But first, a brief methodological aside on what we are doing. Before discussing properties of $\succsim$, we should make explicit what the interpretation of $\succsim$ is. Different methodological stances are possible. Is $\succsim$ tracking what an individual has in mind? What he would say if asked? How he chose in the past?

Under *revealed preference theory*, we interpret $\succsim$ as a description of how an individual **chooses**. Therefore, there is no psychological content to $\succsim$. Revealed preference theory has been the standard methodological stance in economics for a long time. But why? Wouldn't it be better to develop a theory that exploits psychological insights?

Revealed preference theory is a methodological stance, not a psychological (or, for that matter, a moral) one. The assumption is not that choices are unrelated to psychological motives, but that we abstract from these motives and look for patterns in choices directly. There is a strong advantage in doing so: psychological motives are hard to observe, while choices can be observed easily. The implication is that a choice theory based on revealed preferences is empirically testable: if we observe choices that violate the theory's assumptions, we can reject it. Therefore, revealed preference theory is **not** a claim about how individuals make choices or about what drives them. On the contrary, it is deliberately silent about these issues.[1] This is often misunderstood: there is a plethora of critics claiming that economics views individuals as cold robots.[2]

Such critics mostly come from behavioural economics, a field that aims to incorporate psychological insights into economic models.[3] Is it therefore impossible to do behavioural economics within the revealed-preference framework? Not at all. Good behavioural theories do what the name suggests: they characterise the *behavioural* content of a theory, so that we, as economists, can understand how individuals behave. Two behavioural theories with different psychological content but that are observationally equivalent—i.e., they make the same predictions about choices—have the same economic implications.[4]

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[1] If you are interested, see Thoma (2021) for a discussion of the current status of revealed preference theory and Moscati (2025) for the role of psychological narratives in choice theory.

[2] By the way, if you read Asimov's books you know that robots are not cold at all!

[3] See Spiegler (2024) for an account of the motivations of the founding fathers of behavioural economics.

[4] There is a huge debate on this topic. Among many, I suggest reading Gul & Pesendorfer (2008) and the reply by Camerer (2008). A more recent discussion is Spiegler (2019).

An interesting case study is Masatlioglu & Raymond (2016), where the authors show that the famous model by Köszegi & Rabin (2007) is behaviourally equivalent to the intersection of rank-dependent utility and quadratic utility—two older models—despite having a different psychological interpretation. Another example that is quite relevant today is in Eliaz & Spiegler (2006).

In what follows, you can have in mind the interpretation of $\succsim$ that you prefer, but keep in mind that assumptions may have different flavours under different interpretations. Recall that we want to find properties that single out expected utility preferences in Equation (1.1). Therefore, it might be worthwhile to first understand some implications of having expected utility preferences.

Having expected utility preferences over lotteries implies that indifference curves on the simplex are straight lines. That is, say that if $p \sim q$, then, for any $\alpha \in (0,1)$, it holds that $\alpha p + (1-\alpha)q \sim p$, as illustrated in Figure 2.1.

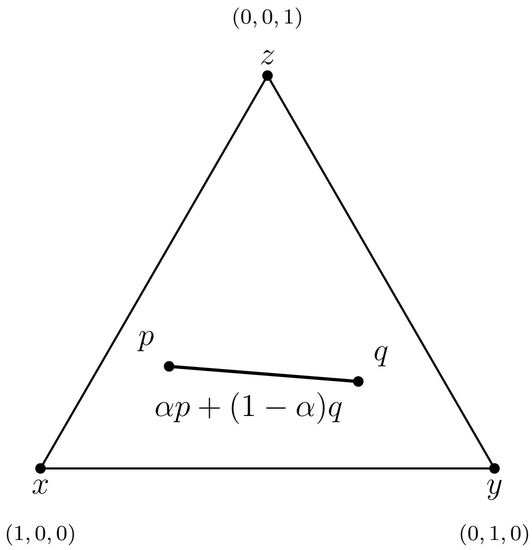

Figure 2.1: If $p \sim q$, then any mixture of p and q is also indifferent to p and q.

Let's show this formally. Assume that $p \sim q$. Then, by the definition of expected utility, we have

$$\sum_{x \in X} p(x)u(x) = \sum_{x \in X} q(x)u(x).$$

Applying expected utility again, for any $\alpha \in (0,1)$, the utility of the lottery $\alpha p + (1-\alpha)q$ is

$$
\begin{aligned}
\sum_{x\in X}\big(\alpha p(x)+(1-\alpha)q(x)\big)u(x) &= \sum_{x\in X}\alpha p(x)u(x)+\sum_{x\in X}(1-\alpha)q(x)u(x)\\
&= \alpha\sum_{x\in X}p(x)u(x)+(1-\alpha)\sum_{x\in X}q(x)u(x)\\
&= \alpha\sum_{x\in X}q(x)u(x)+(1-\alpha)\sum_{x\in X}q(x)u(x)\\
&= \sum_{x\in X}q(x)u(x).
\end{aligned}
$$

Indifference curves are also parallel; you are asked to show this in Exercise 2.1. Of course, the fact that indifference curves are straight lines is related to the linearity of expected utility, which in turn follows from a specific axiom, as we will see shortly.

Let's now turn to the properties of $\succsim$ we will consider. First, we assume that preferences form a **weak order**.

Axiom 2.1. *(**Weak order**) Preferences $\succsim$ are complete and transitive.*

Recall that preferences are **complete** if, for any two lotteries p, q, either $p \succsim q$ or $q \succsim p$, or both. They are **transitive** if, for any three lotteries p, q, r, whenever $p \succsim q$ and $q \succsim r$, then $p \succsim r$.

Sometimes Weak order is referred to as **rationality** of preferences (see e.g. Mas-Colell et al. (1995, p. 6)). However, I think this is an unfortunate name. It suggests that it is "irrational" to violate Weak order, but there are reasons why people might have intransitive or incomplete preferences (can you think of any?). An interesting discussion on the relationship between rationality and *intelligence* is in Myerson (1997, ch. 1).

Weak order is a necessary condition for having any utility representation (see Mas-Colell et al. (1995, p. 9)). It is not the core assumption of expected utility theory, but rather one shared by most theories of choice.

Axiom 2.2. *(**Continuity**) For any three lotteries p, q, r, if $p \succ q \succ r$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha)r \succ q \succ \beta p + (1-\beta)r$.*

Continuity says that there is no lottery p so good that, for $q \succ r$, a small probability β of p and a large probability $1-\beta$ of r is always better than q. Similarly, there is no gamble r so bad that, for $p \succ q$, a large probability α of p and a small probability $1-\alpha$ of r is always worse than q. In essence, this axiom implies that preferences do not have "jumps" when probabilities change slightly—i.e., that preferences are *continuous* in probabilities. Continuity allows us to obtain a continuous utility representation of preferences (see Mas-Colell et al. (1995, p. 47)), but again, it is not the core assumption of expected utility theory—the next one is.

Axiom 2.3. ***(Independence)*** *For any three lotteries p, q, r and for any $\alpha \in (0,1)$, we have $p \succsim q$ if and only if $\alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$.*

In words, Independence says that if p is preferred to q, then mixing both lotteries with any third lottery r, using the same probability $1-\alpha$, does not change their ranking. One way to justify Independence is as follows. Suppose $p \succsim q$. Now consider two compound lotteries obtained by tossing a coin: the first yields p if the coin shows heads and r otherwise; the second yields q if the coin shows heads and r otherwise. Ex ante, one might reason that what happens if the coin shows tails is the same in both compound lotteries, so that part should not matter—while if the coin shows heads, p is preferred to q. Therefore, the first compound lottery should be preferred to the second.

This argument relies on the meaning of the mixing operation within the set of lotteries. By contrast, consider the case of mixing foods. One might prefer pasta to cake, yet mixing both with whipped cream could make the cake better than the pasta.[5] You are asked in Exercise 2.2 to elaborate on the relation between Independence and the linearity of indifference curves.

Before stating Theorem 2.1 in the next section, we need to prove a preliminary result. Lemma 2.1 establishes that, under the assumptions introduced so far, there exist two lotteries that are the best and the worst possible ones.

Lemma 2.1. *Let $\succsim$ satisfy Weak order and Independence. Then there exist two lotteries $\overline{p}$ and $\underline{p}$ such that*

$$\overline{p} \succsim p \succsim \underline{p} \quad \text{for all } p.$$

Proof. The proof proceeds in two steps.

Step 1. By Weak order, the restriction of $\succsim$ to the set of degenerate lotteries $\{\delta_x \in \Delta(X) : \delta_x(x) = 1\}$ is a complete and transitive order on a finite set. Hence there exist outcomes x^*, x_* such that

$$\delta_{x^*} \succsim \delta_x \succsim \delta_{x_*} \qquad \text{for all } x.$$

Fix $\overline{p} := \delta_{x^*}$ and $\underline{p} := \delta_{x_*}$.

Step 2. For any lottery p, let $\text{supp}(p) = \{x \in X : p(x) > 0\}$ and denote its size by $|\text{supp}(p)|$. We prove by induction on $k := |\text{supp}(p)|$ that

$$\overline{p} \succsim p \succsim \underline{p}.$$

Base case. If $k = 1$, then $p = \delta_x$ for some x, and the claim follows from **Step 1**.

Inductive step. Assume the statement holds for all lotteries whose support size is at most $k-1$. Let p have support size $k \geq 2$. Pick any $x \in \text{supp}(p)$ and write

$$p = \alpha\, \delta_x + (1-\alpha)\, q, \qquad \alpha := p(x) \in (0,1),$$

[5] Feel no shame if you are unconvinced by this example.

where q is the renormalized remainder, defined by

$$q(y) = \begin{cases} \dfrac{p(y)}{1-\alpha} & \text{if } y \neq x, \\ 0 & \text{if } y = x. \end{cases}$$

Then $q \in \Delta(X)$ and $|\mathrm{supp}(q)| \leq k-1$.

By the inductive hypothesis, $\overline{p} \succsim q$; and by **Step 1**, $\overline{p} \succsim \delta_x$. We apply Independence twice. From $\overline{p} \succsim q$, mix with $\overline{p}$:

$$\overline{p} = (1-\alpha)\,\overline{p} + \alpha\,\overline{p} \;\succsim\; (1-\alpha)\,q + \alpha\,\overline{p} = \alpha\,\overline{p} + (1-\alpha)\,q.$$

From $\overline{p} \succsim \delta_x$, mix with q:

$$\alpha\,\overline{p} + (1-\alpha)\,q \;\succsim\; \alpha\,\delta_x + (1-\alpha)\,q = p.$$

By transitivity,

$$\overline{p} \succsim p.$$

A symmetric argument yields $p \succsim \underline{p}$. Indeed, by the inductive hypothesis $q \succsim \underline{p}$ and by **Step 1** $\delta_x \succsim \underline{p}$. Using Independence with the same reasoning gives

$$p = \alpha\,\delta_x + (1-\alpha)\,q \;\succsim\; \alpha\,\underline{p} + (1-\alpha)\,q \;\succsim\; \underline{p}.$$

Therefore, $\overline{p} \succsim p \succsim \underline{p}$ for all lotteries with support size k, completing the induction. The fixed degenerate lotteries $\overline{p} = \delta_{x^*}$ and $\underline{p} = \delta_{x_*}$ bound every p, as claimed. □

2.2 Expected utility representation

We are ready to state and prove the theorem relating the properties of preferences over lotteries to the expected utility functional form.

Theorem 2.1. *Preferences over lotteries $\succsim$ satisfy Weak order, Continuity, and Independence if and only if there exists a utility function $u\colon X \to \mathbb{R}$ such that*

$$p \succsim q \text{ if and only if } \sum_{x\in X} p(x)\,u(x) \geq \sum_{x \in X} q(x)\,u(x) \quad \text{for all } p, q. \tag{2.1}$$

The proof essentially follows Mas-Colell et al. (1995, pp. 176–178), complemented by intuition and figures.

Proof. We proceed by steps.

Step 1. If $p \succsim q$, then $p \succsim \alpha p + (1-\alpha)q \succsim q$ for any $\alpha \in (0,1)$.

The intuition behind this step is simple: if p is better than q, then any mixture of the two is worse than p and better than q. Figure 2.2 illustrates the idea.

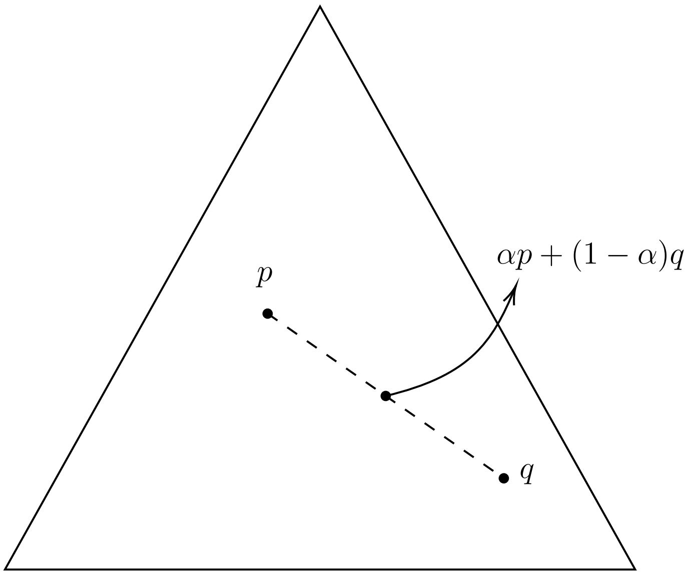

Figure 2.2: Step 1.

This follows from Independence.

$$p \succsim q \implies (1-\alpha)\, p + \alpha\, p \succsim (1-\alpha)\, q + \alpha\, p \implies p \succsim \alpha p + (1-\alpha) q. \tag{2.2}$$

$$p \succsim q \implies \alpha p + (1-\alpha) q \succsim \alpha q + (1-\alpha) q \implies \alpha p + (1-\alpha) q \succsim q. \tag{2.3}$$

The conclusion follows from Equations (2.2) and (2.3).

Step 2. $\beta > \alpha$ if and only if $\beta \overline{p} + (1-\beta)\, \underline{p} \succ \alpha \overline{p} + (1-\alpha)\, \underline{p}$, where $\overline{p}$ and $\underline{p}$ are the best and worst lotteries identified in Lemma 2.1.

The idea of this step is as follows. From **Step 1**, we know that a mixture of p and q, where $p \succsim q$, is worse than p and better than q. Now, since $\overline{p}$ is the best lottery available, we have $\overline{p} \succ \alpha \overline{p} + (1-\alpha)\, \underline{p}$. We want to show that $\beta \overline{p} + (1-\beta)\, \underline{p}$ can be written as a mixture of $\overline{p}$ and $\alpha \overline{p} + (1-\alpha)\, \underline{p}$; therefore, by **Step 1**, it must be preferred to $\alpha \overline{p} + (1-\alpha)\, \underline{p}$. The idea is illustrated in Figure 2.3.

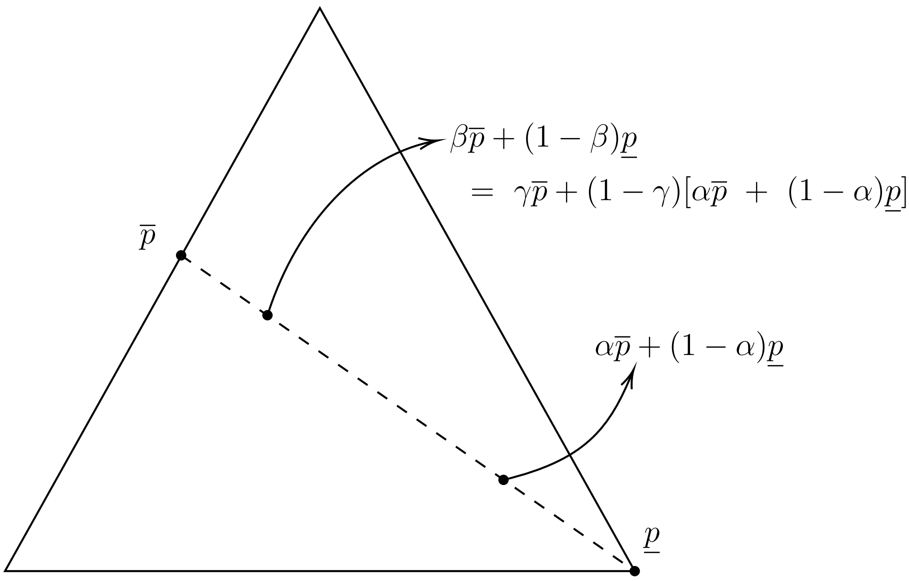

Figure 2.3: Step 2.

We want to express $\beta\overline{p} + (1-\beta)\,\underline{p}$ as a mixture of $\overline{p}$ and $\alpha\overline{p} + (1-\alpha)\,\underline{p}$. That is, we look for some $\gamma \in (0,1)$ such that

$$\beta\overline{p} + (1-\beta)\,\underline{p} = \gamma\overline{p} + (1-\gamma)\big[\alpha\overline{p} + (1-\alpha)\,\underline{p}\big].$$

A short calculation shows that $\gamma = \dfrac{\beta-\alpha}{1-\alpha}$. By **Step 1** we know that $\overline{p} \succ \alpha\overline{p} + (1-\alpha)\,\underline{p}$; therefore,

$$\gamma\overline{p} + (1-\gamma)\big[\alpha\overline{p} + (1-\alpha)\,\underline{p}\big] \succ \alpha\overline{p} + (1-\alpha)\,\underline{p}.$$

Since $\beta\overline{p} + (1-\beta)\,\underline{p} = \gamma\overline{p} + (1-\gamma)\big[\alpha\overline{p} + (1-\alpha)\,\underline{p}\big]$, the conclusion follows.

Up to this point we have proved that if $\beta > \alpha$, then $\beta\overline{p} + (1-\beta)\,\underline{p} \succ \alpha\overline{p} + (1-\alpha)\,\underline{p}$. But the statement says "if and only if", so we must also show the converse: if $\alpha \geq \beta$, then it cannot be that $\beta\overline{p} + (1-\beta)\,\underline{p} \succ \alpha\overline{p} + (1-\alpha)\,\underline{p}$. When $\beta = \alpha$, the two lotteries coincide and are therefore indifferent. The relevant case is $\alpha > \beta$. By the argument above, $\alpha\overline{p} + (1-\alpha)\,\underline{p} \succ \beta\overline{p} + (1-\beta)\,\underline{p}$, and that completes the proof of this step.

Step 3. [6] For any p, there exists a unique $\alpha_p \in [0,1]$ such that $p \sim \alpha_p\overline{p} + (1-\alpha_p)\,\underline{p}$.

We can derive this step as a consequence of the previous ones together with Continuity. This step involves some algebra, but you can get intuition from Figure 2.4.

[6] In this step we use a proof by contradiction. Before diving in, make sure you are familiar with the logic of such proofs.

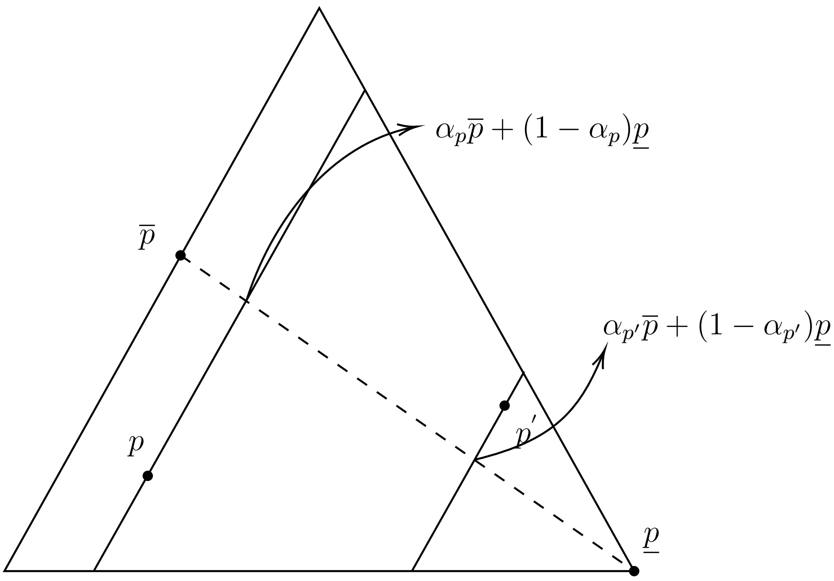

Figure 2.4: Step 3.

First, notice that if α_p exists, it must be unique. Suppose there are two such numbers, α_p and α'_p, with $\alpha_p > \alpha'_p$. Then, by **Step 2**, $\alpha_p\overline{p}+(1-\alpha_p)\,\underline{p} \succ \alpha'_p\overline{p}+(1-\alpha'_p)\,\underline{p}$, contradicting indifference to p.

Now we need to show that such an α_p exists. If $\overline{p} \sim p$, then $\alpha_p = 1$ works; if $\underline{p} \sim p$, then $\alpha_p = 0$ works. The interesting case is when $\overline{p} \succ p \succ \underline{p}$.

Define

$$\alpha_p \;=\; \sup\bigl\{\, \alpha \in [0,1] : p \succsim \alpha\overline{p} + (1-\alpha)\,\underline{p} \,\bigr\}. \tag{2.4}$$

Since $\alpha = 0$ belongs to this set, the supremum is well defined and the set is non-empty.

We now establish two auxiliary claims. The first is

$$\text{If } 1 \geq \alpha > \alpha_p, \text{ then } \alpha\overline{p} + (1-\alpha)\,\underline{p} \succ p. \tag{2.5}$$

Indeed, if $p \succsim \alpha\overline{p}+(1-\alpha)\,\underline{p}$ held for such α, then α_p would not satisfy Equation (2.4). Moreover,

$$\text{If } 0 \leq \alpha < \alpha_p, \text{ then } p \succ \alpha\overline{p} + (1-\alpha)\,\underline{p}. \tag{2.6}$$

The reasoning is as follows. By the definition of α_p, there exists some α' such that $\alpha < \alpha' \leq \alpha_p$ and $p \succsim \alpha'\overline{p} + (1-\alpha')\,\underline{p}$. Since $\alpha < \alpha'$, **Step 2** implies that

$$p \succ \alpha'\overline{p} + (1-\alpha')\,\underline{p} \succ \alpha\overline{p} + (1-\alpha)\,\underline{p}.$$

Now, there are three possibilities to consider: $\alpha_p\overline{p}+(1-\alpha_p)\,\underline{p} \succ p$, $p \succ \alpha_p\overline{p}+(1-\alpha_p)\,\underline{p}$, or indifference between them.

If $\alpha_p\overline{p} + (1-\alpha_p)\,\underline{p} \succ p$, then by Continuity there exists $\beta \in (0,1)$ such that

$$\beta\bigl(\alpha_p\overline{p} + (1-\alpha_p)\,\underline{p}\bigr) + (1-\beta)\,\underline{p} \succ p.$$

Notice that

$$\begin{aligned}\beta\big(\alpha_p\overline{p} + (1-\alpha_p)\,\underline{p}\big) + (1-\beta)\,\underline{p} &= \beta\alpha_p\,\overline{p} + \beta(1-\alpha_p)\,\underline{p} + (1-\beta)\,\underline{p} \\ &= \beta\alpha_p\,\overline{p} + \big[\beta(1-\alpha_p) + (1-\beta)\big]\,\underline{p} \\ &= \beta\alpha_p\,\overline{p} + \big(1-\beta\alpha_p\big)\,\underline{p} \;\succ\; p.\end{aligned}$$

Since $\beta\alpha_p < \alpha_p$, by Equation (2.6) we must have $p \succ \beta\alpha_p\,\overline{p} + (1-\beta\alpha_p)\,\underline{p}$, which is a contradiction.

If instead $p \succ \alpha_p\overline{p} + (1-\alpha_p)\,\underline{p}$, then by Continuity there exists $\beta \in (0,1)$ such that

$$\begin{aligned} p &\succ \beta\big(\alpha_p\overline{p} + (1-\alpha_p)\,\underline{p}\big) + (1-\beta)\,\overline{p} \\ &= \big[\beta\alpha_p + (1-\beta)\big]\,\overline{p} + \beta(1-\alpha_p)\,\underline{p} \\ &= \big(1-\beta(1-\alpha_p)\big)\,\overline{p} + \beta(1-\alpha_p)\,\underline{p}.\end{aligned}$$

Since $1-\beta(1-\alpha_p) > \alpha_p$, by Equation (2.5) we must have

$$\big(1-\beta(1-\alpha_p)\big)\,\overline{p} + \beta(1-\alpha_p)\,\underline{p} \succ p,$$

which is again a contradiction.

Step 4. Define a utility function $U\colon \Delta(X) \to \mathbb{R}$ that assigns to each lottery a number representing its utility, defined by $U(p) = \alpha_p$. This function represents preferences $\succsim$.

Take two lotteries p and p'. By **Step 3**, there exist unique α_p and $\alpha_{p'}$ such that

$$p \sim \alpha_p\overline{p} + (1-\alpha_p)\,\underline{p}, \quad p' \sim \alpha_{p'}\overline{p} + (1-\alpha_{p'})\,\underline{p}.$$

Therefore,

$$p \succsim p' \text{ if and only if } \alpha_p\overline{p} + (1-\alpha_p)\,\underline{p} \succsim \alpha_{p'}\overline{p} + (1-\alpha_{p'})\,\underline{p}.$$

By **Step 2**,

$$\alpha_p\overline{p} + (1-\alpha_p)\,\underline{p} \succsim \alpha_{p'}\overline{p} + (1-\alpha_{p'})\,\underline{p} \text{ if and only if } \alpha_p \geq \alpha_{p'}.$$

The last condition holds if and only if $U(p) \geq U(p')$, which proves the claim.

Step 5. The function U is linear and therefore, by Exercise 1.6, has the expected utility form.

From the previous steps we know that, for any lottery p, there is a unique number $U(p) \in [0,1]$ such that

$$p \sim U(p)\,\overline{p} + \big(1-U(p)\big)\,\underline{p}, \qquad p' \sim U(p')\,\overline{p} + \big(1-U(p')\big)\,\underline{p}.$$

Applying Independence, we get

$$\begin{aligned}\beta p + (1-\beta)p' &\sim \beta\big[U(p)\,\overline{p} + \big(1-U(p)\big)\,\underline{p}\big] + (1-\beta)p' \\ &\sim \beta\big[U(p)\,\overline{p} + \big(1-U(p)\big)\,\underline{p}\big] + (1-\beta)\big[U(p')\,\overline{p} + \big(1-U(p')\big)\,\underline{p}\big] \\ &= \big[\beta U(p) + (1-\beta)U(p')\big]\,\overline{p} + \Big(1 - \big[\beta U(p) + (1-\beta)U(p')\big]\Big)\,\underline{p}.\end{aligned}$$

Let $\gamma := \beta U(p) + (1-\beta)U(p')$. By **Step 4**, for the lottery $\beta p + (1-\beta)p'$ there is a *unique* number γ such that $\beta p + (1-\beta)p' \sim \gamma\,\overline{p} + (1-\gamma)\,\underline{p}$. Therefore,

$$U(\beta p + (1-\beta)p') = \beta U(p) + (1-\beta)U(p').$$

$\square$

Recall that a functional representation of preferences need not be unique: multiple functions can represent the same preferences. However, for expected utility representations we have a very specific characterization of all possible representations, as stated in the following corollary.

Corollary 2.1. *Suppose U is an expected utility representation of $\succsim$. Then $\widetilde{U}\colon \Delta(X) \to \mathbb{R}$ is another expected utility representation of $\succsim$ if and only if there exist $\beta > 0$ and $\gamma \in \mathbb{R}$ such that*

$$\widetilde{U}(p) \;=\; \beta\, U(p) + \gamma \qquad \textit{for all } p. \tag{2.7}$$

Proof. First, suppose that Equation (2.7) holds. Then $\widetilde{U}$ is an expected utility representation of $\succsim$. Assume $\widetilde{U} = \beta U + \gamma$ with $\beta > 0$. Then

$$\widetilde{U}(p) \;=\; \beta \sum_x p(x)u(x) + \gamma \;=\; \sum_x p(x)\big[\beta u(x) + \gamma\big].$$

Hence, $\widetilde{U}$ has the expected utility form with $\widetilde{u}(x) := \beta u(x) + \gamma$. Since $\beta > 0$, it follows that $p \succsim q \iff U(p) \geq U(q) \iff \widetilde{U}(p) \geq \widetilde{U}(q)$.

Second, suppose that U and $\widetilde{U}$ are both expected utility representations of the same $\succsim$. Then they must be related by an affine transformation as in Equation (2.7). By Lemma 2.1, there exist $\overline{p}, \underline{p}$ such that $\overline{p} \succ \underline{p}$ and $\overline{p} \succsim p \succsim \underline{p}$ for all p. For any lottery p, define $\alpha_p \in [0,1]$ by

$$U(p) \;=\; \alpha_p U(\overline{p}) + (1-\alpha_p)U(\underline{p}), \quad \text{so that} \quad \alpha_p \;=\; \frac{U(p) - U(\underline{p})}{U(\overline{p}) - U(\underline{p})}.$$

Applying the same construction to $\widetilde{U}$ and using the *same* α_p (since both functions represent the *same* preferences), we obtain

$$\widetilde{U}(p) \;=\; \alpha_p \widetilde{U}(\overline{p}) + (1-\alpha_p)\,\widetilde{U}(\underline{p}) \;=\; \widetilde{U}(\underline{p}) \;+\; \frac{\widetilde{U}(\overline{p}) - \widetilde{U}(\underline{p})}{U(\overline{p}) - U(\underline{p})}\big[U(p) - U(\underline{p})\big].$$

Rearranging yields the affine relation

$$\widetilde{U}(p) \;=\; \beta\, U(p) + \gamma \quad \text{for all } p,$$

where

$$\beta \;:=\; \frac{\widetilde{U}(\overline{p}) - \widetilde{U}(\underline{p})}{U(\overline{p}) - U(\underline{p})} \;>\; 0, \qquad \gamma \;:=\; \widetilde{U}(\underline{p}) - \beta\, U(\underline{p}).$$

Positivity of β follows because $\overline{p} \succ \underline{p}$ implies $U(\overline{p}) > U(\underline{p})$ and $\widetilde{U}(\overline{p}) > \widetilde{U}(\underline{p})$. $\square$

In Exercise 2.7, you are asked to elaborate on the significance of Corollary 2.1 for interpreting utility numbers as *cardinal* representations of preferences.[7]

Things to read. For a textbook treatment of the material covered in this lecture, see Mas-Colell et al. (1995, pp. 170–178). Theorem 2.1 is known as the *von Neumann–Morgenstern representation theorem*, after von Neumann & Morgenstern (2007), and it has enormous historical importance. An excellent discussion of the historical context and significance of expected utility theory can be found in Moscati (2018).

2.3 Exercises

Exercise 2.1. Show that if preferences are represented by an expected utility function, then indifference curves in the triangle are parallel lines.

Solution to Exercise 2.1. Check Mas-Colell et al. (1995, p. 175). □

Exercise 2.2. Explain why the fact that indifference curves are straight and parallel lines follows from Independence. I.e., use the axiom, not the functional form!

Solution to Exercise 2.2. Check Mas-Colell et al. (1995, p. 175-176). □

Exercise 2.3. Prove the direction of Theorem 2.1 that we did not prove in class. Show that if U represents $\succsim$, then $\succsim$ satisfies Weak order, Continuity, and Independence. (It is not difficult, I promise!)

Solution to Exercise 2.3. Weak order. Since $\mathbb{R}$ with the usual order is a weak order, and U represents $\succsim$, it follows that $\succsim$ is a weak order.

Continuity. Take three lotteries p, q, r such that $p \succsim q \succsim r$. Then,

$$U(p) \geq U(q) \geq U(r).$$

Since U is an expected utility function, it is continuous (as a function from a finite-dimensional Euclidean space to $\mathbb{R}$). Therefore, for any sequence of lotteries $(q_n)_{n=1}^{\infty}$ converging to q, we have

$$\lim_{n\to\infty} U(q_n) \;=\; U(q).$$

If $q_n \to q$ and $p \succsim q_n$ for all n, then

$$U(p) \;\geq\; U(q_n) \;\to\; U(q),$$

implying $p \succsim q$. A similar argument shows the other part of Continuity.

[7] If you are interested in this aspect, you may read von Neumann & Morgenstern (2007, ch. 3). An illuminating discussion of measurement in the social sciences is in Krantz et al. (1971, ch. 1).

Independence. Take three lotteries p, q, r such that $p \succsim q$ and any $\alpha \in (0,1)$. Then,

$$U(p) \;\geq\; U(q).$$

Since U is an expected utility function, we have

$$U(\alpha p + (1-\alpha)r) \;=\; \alpha U(p) + (1-\alpha)U(r), \quad U(\alpha q + (1-\alpha)r) \;=\; \alpha U(q) + (1-\alpha)U(r).$$

It follows that

$$U(\alpha p + (1-\alpha)r) \;\geq\; U(\alpha q + (1-\alpha)r).$$

Since U represents $\succsim$, this implies

$$\alpha p + (1-\alpha)r \;\succsim\; \alpha q + (1-\alpha)r,$$

which is exactly Independence. □

Exercise 2.4. Revisit the choice problem in Exercise 1.7. Show that the preferences exhibited there do not satisfy Independence. Representing the lotteries in a table might help.

Solution to Exercise 2.4. Check Mas-Colell et al. (1995, p. 179-180). □

Exercise 2.5. Consider the **Betweenness Axiom** introduced by Dekel (1986): for all lotteries p, q and $\alpha \in [0,1]$, if $p \sim q$, then $\alpha p + (1-\alpha)q \sim p$. Show that Independence implies Betweenness, but Betweenness does not imply Independence. (Hint: for the second part, construct a preference relation that satisfies Betweenness but not Independence, maybe in the graph.) Are indifference curves still linear under Betweenness? Are they parallel?

Solution to Exercise 2.5. Independence implies Betweenness. Take any lotteries p, q such that $p \sim q$ and any $\alpha \in [0,1]$. By Independence,

$$\alpha p + (1-\alpha)q \;\sim\; \alpha p + (1-\alpha)p \;=\; p.$$

Betweenness does not imply Independence. Consider the following preference relation over the triangle with vertices x, y, z:

- All lotteries on the line segment between x and y are indifferent to each other.
- All lotteries on the line segment between y and z are indifferent to each other.
- All lotteries on the line segment between z and x are indifferent to each other.
- Any lottery on one side of the triangle is strictly preferred to any lottery on the other side of the triangle.

This preference relation satisfies Betweenness but not Independence. For example, take p on the line segment between x and y, and q on the line segment between y and z, so that $p \sim q$. Now take $r = z$. Then, for any $\alpha \in (0,1)$,

$$\alpha p + (1-\alpha) r \succ \alpha q + (1-\alpha) r,$$

showing that Independence fails.

Under Betweenness, indifference curves are still linear, but they need not be parallel. □

Exercise 2.6. Show that the choices of the individual in Exercise 1.7 are compatible with Betweenness. (Hint: drawing a picture might help.)

Exercise 2.7. Explain why Corollary 2.1 allows us to make statements such as "the difference in utility between lottery p and lottery q is greater than the difference in utility between lottery r and lottery s". Why this would not be possible without Corollary 2.1?

Solution to Exercise 2.7. Corollary 2.1 states that any two expected utility representations of the same preference relation are related by an affine transformation. Therefore, if U and $\widetilde{U}$ are two expected utility representations of the same preferences, then for any lotteries p, q, r, s,

$$U(p) - U(q) \;>\; U(r) - U(s) \quad \text{if and only if} \quad \widetilde{U}(p) - \widetilde{U}(q) \;>\; \widetilde{U}(r) - \widetilde{U}(s).$$

This is because

$$\begin{aligned} \widetilde{U}(p) - \widetilde{U}(q) &= \beta\, U(p) + \gamma - \big(\beta\, U(q) + \gamma\big) \\ &= \beta\big[U(p) - U(q)\big], \end{aligned}$$

and similarly for r and s. Since $\beta > 0$, the inequalities are preserved.

Without Corollary 2.1, we could not guarantee that differences in utility values are meaningful across different expected utility representations. Different representations could yield different rankings of differences in utility, making such statements ambiguous. □

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Lecture 3

Money lotteries

3.1 Structuring the set of outcomes

In the previous section, we studied preferences with the expected utility form over lotteries on a *finite* outcome set X. We now study a setting where the outcome set is the set of real numbers $\mathbb{R}$, representing monetary outcomes. This setting is particularly important in economics and finance, as it allows us to model decisions such as investments, insurance, and consumption.

You may wonder whether a form of Theorem 2.1 extends to this setting. The answer is yes, see Kreps (1988, pp. 59–78) or Fishburn (1970, ch. 10).

Since the outcome set is now infinite, we should be careful about how we define lotteries. We will use cumulative distribution functions (CDFs) to represent lotteries over monetary outcomes. A CDF $F\colon \mathbb{R} \to [0,1]$ maps each monetary outcome x to the probability that the outcome is less than or equal to x. It satisfies:

- F is nondecreasing, i.e. if $x \leq y$, then $F(x) \leq F(y)$.
- $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$.
- F is right–continuous, i.e. for every $x \in \mathbb{R}$, $\lim_{y\downarrow x} F(y) = F(x)$.[1]

Example 3.1. Consider a lottery that pays 1 dollar with probability $\frac{1}{4}$, 4 dollars with probability $\frac{1}{2}$, and 6 dollars with probability $\frac{1}{4}$. The corresponding CDF F is

$$F(x) = \begin{cases} 0 & \text{if } x < 1, \\ \frac{1}{4} & \text{if } 1 \leq x < 4, \\ \frac{3}{4} & \text{if } 4 \leq x < 6, \\ 1 & \text{if } x \geq 6, \end{cases}$$

and it is represented in Figure 3.1.

[1] The notation $y \downarrow x$ means that y approaches x from above.

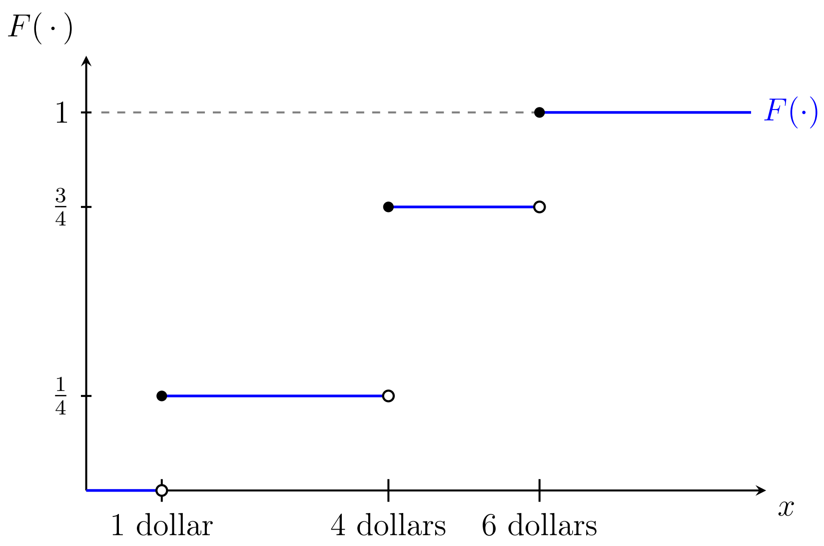

Figure 3.1: Cumulative distribution function (CDF) representing a lottery over monetary outcomes.

■

A second approach to studying lotteries over $\mathbb{R}$ is via *simple* probability distributions, i.e. probability distributions that assign positive probability to only a finite number of outcomes.

Notice that mixtures of CDFs are also CDFs, so we can employ the same mixture operation defined in Section 1.1. In particular, given two CDFs F and G, and $\alpha \in [0,1]$, the mixture $H = \alpha F + (1-\alpha)G$ is also a CDF, where $H(x) = \alpha F(x) + (1-\alpha)G(x)$ for all x.

We now define preferences $\succsim$ over the set of CDFs on $\mathbb{R}$ that have the expected utility form. The idea is the same as before: we weight the utility of each monetary outcome by its probability and sum these weighted utilities to obtain the expected utility of the lottery. A preference relation $\succsim$ over the set of CDFs has the expected utility form if there exists a utility function $u\colon \mathbb{R} \to \mathbb{R}$ such that for any two CDFs F and G,

$$F \succsim G \iff \int u(x)\,\mathrm{d}F(x) \geq \int u(x)\,\mathrm{d}G(x).$$

Earlier, the Bernoulli utility was defined on a finite X as $u\colon X \to \mathbb{R}$; now the outcome set is $\mathbb{R}$, hence the domain differs. This lets us impose properties of u that are specific to monetary outcomes. From now on we assume the following two.

Definition 3.1. *The utility function u is **increasing** if for any x, y with $x > y$, we have $u(x) > u(y)$.*

Definition 3.1 captures the idea that more money is preferred to less. When the outcome set was a generic X, the inequality $x > y$ had no meaning.[2]

[2]For instance, if x is an apple and y is a banana, what would $x > y$ even mean?

Definition 3.2. *The utility function $u\colon \mathbb{R} \to \mathbb{R}$ is* ***continuous*** *if for any x and any $\varepsilon > 0$, there exists $\delta > 0$ such that for all y with $|x - y| < \delta$, we have $|u(x) - u(y)| < \varepsilon$.*

Definition 3.2 ensures that small changes in money lead to small changes in the Bernoulli utility u. This could not be stated with a generic outcome set, where expressions like $x - y$ are undefined.

Definition 3.2 is continuity in *money*. What about continuity in *probabilities*?

3.2 Risk aversion

We now have the tools to define and discuss *risk aversion*. Defining this concept allows us to answer questions such as: does an individual dislike risk? how much? As we will see in the next Lecture, the answer has important implications for economic behaviour.

The definition of risk aversion is quite intuitive. Consider the following choice: receive 5 euros for sure, or take a lottery that pays 0 euros with probability 0.5 and 10 euros with probability 0.5. Both options have the same expected monetary value, 5 euros. If the individual prefers the certain 5 to the lottery, he *dislikes* risk—he prefers getting the mean outcome for sure rather than facing uncertainty. If instead he prefers the lottery, he *likes* risk—he is willing to face uncertainty for the chance of a higher payoff.

Did you notice what we just did? We needed to develop a definition of an intuitive, but ex-ante vague concept, risk aversion. We did it by developing a simple thought experiment that "keeps everything fixed" except for the presence of risk. These thought experiments are very useful to develop effective definitions.

For a general lottery, we say an individual is risk averse if he prefers the certain amount equal to the lottery's expected value to the lottery itself. For a CDF F, the expected value is

$$\int x \, \mathrm{d}F(x). \tag{3.1}$$

Evaluating money through u, the certain amount equal to that expected value yields utility

$$u\left(\int x \, \mathrm{d}F(x)\right), \tag{3.2}$$

whereas the lottery yields expected utility

$$\int u(x) \, \mathrm{d}F(x). \tag{3.3}$$

Definition 3.3. *An individual with expected utility preferences and Bernoulli utility u is* ***risk averse*** *if for each CDF F,*

$$u\Big(\int x \,\mathrm{d}F(x)\Big) \;\geq\; \int u(x)\,\mathrm{d}F(x). \tag{3.4}$$

If u is not increasing (Definition 3.1), Definition 3.3 loses its intended meaning.

Inequality (3.4) is precisely Jensen's inequality and is equivalent to the *concavity* of u. Thus, the intuitive notion of risk aversion is equivalent to concavity of u. If u is twice differentiable, concavity means $u''(x) \leq 0$ for all x; see Figure 3.2.

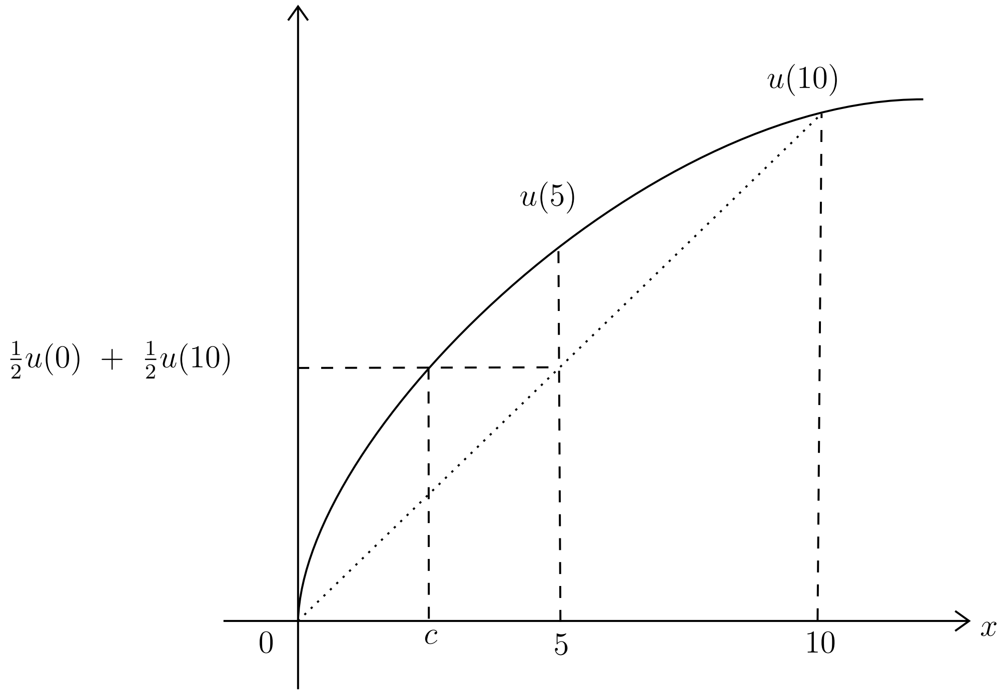

Figure 3.2: Example of a concave u.

Analogously, an individual is *risk loving* if (3.4) is reversed, and *risk neutral* if it holds with equality.

There are other, equivalent ways to define risk aversion. This is good news: it means the definition is robust. One convenient route is via the **certainty equivalent**—the sure amount of money that makes the individual indifferent to the lottery.

Definition 3.4. *The* ***certainty equivalent*** *of a lottery with CDF F for an individual with utility u is the number $c(F,u)$ solving*

$$u\big(c(F,u)\big) = \int u(x)\,\mathrm{d}F(x). \tag{3.5}$$

As an illustration, the certainty equivalent of the lottery paying 0 with probability 1/2 and 10 with probability 1/2 in Figure 3.2 is c.

Intuitively, if an individual is risk averse, his certainty equivalent must be less than the expected value of the lottery, as he prefers receiving the expected value for sure rather than facing the lottery. To capture this intuition we can define the **risk premium** of a lottery as the difference between the expected value of the lottery and its certainty equivalent.

Definition 3.5. *The **risk premium** of a lottery with CDF F for utility u is*

$$\pi(F, u) = \int x \, \mathrm{d}F(x) - c(F, u). \tag{3.6}$$

You will show in Exercise 3.3 that an individual is risk averse if and only if the risk premium is nonnegative for every lottery.

We now have a notion of risk aversion, but not a way to compare the risk attitudes of two individuals. Again, we start from intuition, how could we compare two risk averse individuals? The risk premium might be a starting point, the higher the risk premium, the more risk averse the individual, as he requires a lower certainty equivalent to face the lottery. Consider two individuals with utility function u and v. If for each lottery F, the risk premium of the first individual is higher than that of the second, i.e. $\pi(F, u) \geq \pi(F, v)$, we can say that the first individual is more risk averse than the second. However, such condition boils down to comparing certainty equivalents:

$$\pi(F, u) \geq \pi(F, v) \quad \iff \quad c(F, u) \leq c(F, v),$$

you should show this. We therefore have the following definition.

Definition 3.6. *An individual with utility u is **more risk averse** than one with utility v if, for every lottery F,*

$$c(F, u) \leq c(F, v).$$

We now want to develop a measure of risk aversion that is related to the rate at which the certainty equivalent changes as we change the lottery. Consider a lottery over monetary outcomes that pays $x + \varepsilon$ with probability $1/2$ and $x - \varepsilon$ with probability $1/2$, call it F_ε. By Definition 3.4

$$u\big(c(F_\varepsilon, u)\big) = \tfrac{1}{2}u(x + \varepsilon) + \tfrac{1}{2}u(x - \varepsilon). \tag{3.7}$$

Since both sides of Equation (3.7) are twice differentiable in ε and $u'(c(F_\varepsilon, u)) > 0$ since u is increasing, the implicit function theorem implies that $c(F_\varepsilon, u)$ is twice differentiable in a neighborhood of 0. Differentiating (3.7) with respect to ε gives

$$u'\big(c(F_\varepsilon, u)\big)\, c'(\varepsilon) = \tfrac{1}{2}u'(x + \varepsilon) - \tfrac{1}{2}u'(x - \varepsilon).$$

Evaluating at $\varepsilon = 0$,

$$u'(x)\, c'(0) = 0 \quad \Longrightarrow \quad c'(0) = 0, \qquad c(0) = x.$$

Differentiating again with respect to ε,

$$u''\big(c(F_\varepsilon, u)\big)\big(c'(\varepsilon)\big)^2 + u'\big(c(F_\varepsilon, u)\big)c''(\varepsilon) = \tfrac{1}{2}u''(x+\varepsilon) + \tfrac{1}{2}u''(x-\varepsilon).$$

Evaluating at $\varepsilon = 0$ and using $c'(0) = 0$ and $c(0) = x$, we obtain

$$u'(x)c''(0) = u''(x) \quad \Longrightarrow \quad c''(0) = \frac{u''(x)}{u'(x)}.$$

The ratio between the second and first derivative of the utility function is the **Arrow-Pratt coefficient of absolute risk aversion**. It is not by chance that it appears here. As we noticed already, risk aversion is related to the concavity of the utility function, which is captured by its second derivative. In principle we could use the second derivative alone as a measure of risk aversion, but this would not be satisfactory, as multiplying the utility function by a positive constant would change the second derivative but not risk aversion. Dividing the second derivative by the first derivative solves this problem, as multiplying the utility function by a positive constant multiplies both derivatives by the same constant, leaving their ratio unchanged.

Definition 3.7. *The* ***Arrow-Pratt coefficient of absolute risk aversion*** *for an individual with utility function u at outcome x is*

$$r(x,u) = -\frac{u''(x)}{u'(x)}.$$

Hence, we just showed that the limit of the second derivative of the certainty equivalent as $\varepsilon \to 0$ is exactly $-r(x,u)$. You should notice that, if u is increasing and concave, then $r(x,u)$ is positive. In the exercises, you are asked to show the following equivalence between certainty equivalents and the Arrow-Pratt coefficient.

Proposition 3.1. *An individual with utility u is* ***more risk averse*** *than an individual with utility v if and only if, for each x,*

$$r(x,u) \;\geq\; r(x,v).$$

You should notice that, "more risk averse then" is a partial order on the set of utility functions, it is not complete. There might be two utility functions u and v such that neither u is more risk averse than v, nor v is more risk averse than u. This happens when the Arrow-Pratt coefficients cross, i.e. there exist x and y such that $r(x,u) > r(x,v)$ and $r(y,u) < r(y,v)$.

Things to read. This section mostly draws from Mas-Colell et al. (1995, ch. 6.C.). Alternatives treatments can be found in Kreps (1988, ch. 6) and Kreps (2013, ch. 6).

3.3 Exercises

Exercise 3.1. Check that the CDF in Figure 3.1 satisfies the three properties of a CDF.

Exercise 3.2. Write the condition for having preferences increasing in money in terms of the binary relation $\succsim$ over CDFs. Show that such condition implies that the Bernoulli utility u is increasing.

Solution to Exercise 3.2. Preferences are increasing in money if for any two CDFs F and G such that $F(x) \leq G(x)$ for all x with strict inequality for some x, we have $F \succ G$. To show that this implies that u is increasing, consider two monetary outcomes $x > y$ and the corresponding degenerate CDFs F_x and F_y that assign probability 1 to x and y, respectively. Since $x > y$, we have $F_x(z) \leq F_y(z)$ for all z, with strict inequality at $z = x$. By the assumption of increasing preferences, it follows that $F_x \succ F_y$. According to the expected utility representation, this means:

$$\int u(z)\, dF_x(z) > \int u(z)\, dF_y(z),$$

which simplifies to:

$$u(x) > u(y).$$

Hence, we conclude that the Bernoulli utility function u is increasing. □

Exercise 3.3. Show that, if an individual with expected utility preferences and utility u is risk averse, his risk premium is nonnegative for each lottery.

Solution to Exercise 3.3. By Definition 3.3, an individual is risk averse if for every CDF F,

$$u\Big(\int x\, dF(x)\Big) \geq \int u(x)\, dF(x).$$

Let $c(F, u)$ be the certainty equivalent of the lottery with CDF F. By Definition 3.4, we have:

$$u\big(c(F, u)\big) = \int u(x)\, dF(x).$$

Combining these two equations, we get:

$$u\Big(\int x\, dF(x)\Big) \geq u\big(c(F, u)\big).$$

Since u is increasing (by Definition 3.1), we can apply the monotonicity property of u to the above inequality, yielding:

$$\int x\, dF(x) \geq c(F, u).$$

Rearranging this gives us the risk premium:

$$\pi(F, u) = \int x\, dF(x) - c(F, u) \geq 0.$$

Hence, we conclude that if an individual is risk averse, his risk premium is nonnegative for every lottery. □

Exercise 3.4. We noted that risk aversion is linked to concavity. Can you define "more risk averse than" directly via concavity? Define when a function is "more concave than" another, and show that your notion is equivalent to Definition 3.6. (Hint: it is easiest to go via the Arrow–Pratt coefficient.)

Solution to Exercise 3.4. Check Mas-Colell et al. (1995, p. 190). □

Exercise 3.5. Prove Proposition 3.1. (If you are stuck, check exercises 6.C.6 and 6.C.7 in Mas-Colell et al. (1995).)

Solution to Exercise 3.5. To prove Proposition 3.1, we need to show that an individual with utility u is more risk averse than an individual with utility v if and only if, for each x,

$$r(x,u) \geq r(x,v),$$

where $r(x,u) = -\frac{u''(x)}{u'(x)}$ and $r(x,v) = -\frac{v''(x)}{v'(x)}$.

($\Rightarrow$) Assume that the individual with utility u is more risk averse than the individual with utility v. By Definition 3.6, this means that for every lottery F,

$$c(F,u) \leq c(F,v).$$

Consider a small perturbation of a lottery around a certain outcome x. Specifically, consider the lottery F_ε that pays $x+\varepsilon$ with probability $1/2$ and $x-\varepsilon$ with probability $1/2$. The certainty equivalents for this lottery are given by:

$$u\big(c(F_\varepsilon,u)\big) = \tfrac{1}{2}u(x+\varepsilon) + \tfrac{1}{2}u(x-\varepsilon),$$

and

$$v\big(c(F_\varepsilon,v)\big) = \tfrac{1}{2}v(x+\varepsilon) + \tfrac{1}{2}v(x-\varepsilon).$$

Differentiating both sides with respect to ε and evaluating at $\varepsilon = 0$, we find that:

$$c''(0,u) = -\frac{u''(x)}{u'(x)}, \quad c''(0,v) = -\frac{v''(x)}{v'(x)}.$$

Since $c(F_\varepsilon,u) \leq c(F_\varepsilon,v)$, it follows that:

$$c''(0,u) \leq c''(0,v),$$

implying that:

$$-\frac{u''(x)}{u'(x)} \geq -\frac{v''(x)}{v'(x)},$$

which gives us $r(x,u) \geq r(x,v)$.

($\Leftarrow$) Now, assume that for each x,

$$r(x,u) \geq r(x,v).$$

We need to show that for every lottery F,

$$c(F, u) \leq c(F, v).$$

Consider again the lottery F_ε defined above. Using the same differentiation process, we find that:

$$c''(0, u) = -\frac{u''(x)}{u'(x)}, \quad c''(0, v) = -\frac{v''(x)}{v'(x)}.$$

Since $r(x, u) \geq r(x, v)$, it follows that:

$$c''(0, u) \leq c''(0, v).$$

Integrating this inequality twice with respect to ε and using the fact that $c(0, u) = c(0, v) = x$ and $c'(0, u) = c'(0, v) = 0$, we conclude that:

$$c(F_\varepsilon, u) \leq c(F_\varepsilon, v).$$

Since this holds for any small perturbation ε, it extends to all lotteries F. Thus, we have shown that the individual with utility u is more risk averse than the individual with utility v. □

Exercise 3.6. A specific utility function that is often used in economics and finance is the **exponential utility function**, defined as $u(x) = 1 - e^{-\alpha x}$, where $\alpha > 0$ is a parameter. Such function has an interesting property related to how it handles risk. Can you find it? Can you elaborate on what this property implies for risk taking at different wealth levels?

Solution to Exercise 3.6. To analyze the properties of the exponential utility function $u(x) = 1 - e^{-\alpha x}$, we first compute its first and second derivatives:

$$u'(x) = \alpha e^{-\alpha x},$$

$$u''(x) = -\alpha^2 e^{-\alpha x}.$$

Next, we calculate the Arrow-Pratt coefficient of absolute risk aversion:

$$r(x, u) = -\frac{u''(x)}{u'(x)} = -\frac{-\alpha^2 e^{-\alpha x}}{\alpha e^{-\alpha x}} = \alpha.$$

This result shows that the Arrow-Pratt coefficient of absolute risk aversion $r(x, u)$ is constant and equal to α, regardless of the wealth level x.

The implication of this property is that individuals with exponential utility exhibit **constant absolute risk aversion** (CARA). This means that their willingness to take risks does not change with their wealth level. In other words, an individual with exponential utility will require the same risk premium to accept a risky lottery, regardless of whether they are wealthy or poor. □

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Lecture 4

Stochastic dominance and applications

4.1 Stochastic dominance

In Lecture 3, we developed an analysis of properties of expected utility preferences. However, we have not yet discussed relevant properties of lotteries, which is what we do now.

To start, one might want to have a language to say that a “lottery yields higher returns than another one”. A simple way of capturing this idea is the following: an individual with expected utility preferences should prefer lottery F to lottery G for any possibile utility function u. This is first-order stochastic dominance.

Definition 4.1. *The lottery F* ***first-order stochastically dominates*** *G if*

$$\int u(x)dF(x) \geq \int u(x)dG(x) \quad \textit{for every nondecreasing } u. \tag{4.1}$$

There is a second way of capturing the idea: for each given return, the probability of getting at least that return is higher in one lottery than in the other one. That is, for each return x, if $F(x) \leq G(x)$, then the probability of getting at least x is higher in lottery F than in lottery G, because the probability of getting at least x in lottery F is $1 - F(x)$. The two criteria (4.1) and $F(x) \leq G(x)$ are equivalent, as stated by the following result.

Proposition 4.1. *Lottery F first-order stochastically dominates lottery G if and only if $F(x) \leq G(x)$.*

You are asked to prove one direction of Proposition 4.1 in Exercise 4.1. As illustrated in Figure 4.1, lottery F first-order stochastically dominates lottery G if its graph is always below the graph of lottery G.

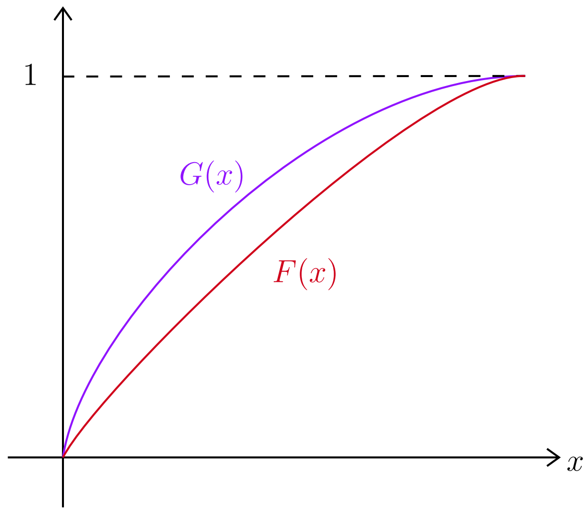

Figure 4.1: Lottery F first-order stochastically dominates lottery G.

Notice that first-order stochastic dominance is an *incomplete* ordering over lotteries: there are pairs of lotteries F, G such that neither F first-order stochastically dominates G nor G first-order stochastically dominates F.

First order stochastic dominance is a comparison of returns. We now develop a notion allowing us to compare riskiness. Again, we start from an intuitive idea: say that two lotteries have the same expected return, but a risk averter prefers one lottery to the other. Since the individual is risk averse, she must be preferring the less risky lottery. In this case, we say that the first lottery second-order stochastically dominates the second one.

Definition 4.2. *The lottery F* ***second-order stochastically dominates*** *G with the same mean if*

$$\int u(x)dF(x) \geq \int u(x)dG(x) \quad \textit{for every nondecreasing concave } u.$$

Recall that if an expected utility maximiser has a concave utility function, he is risk averse, which explains the qualifier in Definition 4.2.

There is a second intuitive way of defining second-order stochastic dominance using the concept of a *mean preserving spread*. Consider the following compound lottery. First, an outcome x is drawn according to a distribution F. Then, to the realisation x, an amount z, distributed according to a distribution with mean zero, is added. The resulting lottery has the same mean as F but is more spread out, hence riskier. Such a lottery is called a mean preserving spread of F. If a lottery G can be constructed in this way from lottery F, we say that G is a mean preserving spread of F. We have the following result.

Proposition 4.2. *If two distributions F have the same mean G, then F second-order stochastically dominates G if and only if G is a mean preserving spread of F.*

4.2 Applications

We conclude our treatment of (objective) expected utility theory with two applications of the concepts developed so far: insurance and investment in a risky asset.

Insurance. Consider an expected utility maximiser with initial wealth $w > 0$ who faces a possible loss $D > 0$. The loss occurs with probability $\pi \in (0,1)$ and does not occur with probability $1-\pi$. It is possible to buy insurance: one unit of insurance costs q euros for sure and pays 1 euro if the loss occurs.

If the individual buys $\alpha \geq 0$ units of insurance, his decision problem is

$$\max_{\alpha \geq 0} \Big\{ (1-\pi)\, u(w - \alpha q) + \pi\, u(w - D - \alpha q + \alpha) \Big\}. \tag{4.2}$$

Assume an interior optimum $\alpha^* > 0$. Differentiating Equation (4.2) with respect to α gives

$$(1-\pi)\, u'(w - \alpha q)(-q) + \pi\, u'(w - D - \alpha q + \alpha)\,(1-q),$$

so α^* satisfies

$$(1-\pi)\, u'(w - \alpha^* q)(-q) + \pi\, u'(w - D - \alpha^* q + \alpha^*)\,(1-q) = 0.$$

If the individual is risk neutral, and therefore u is linear, the individual maximises expected wealth:

$$w - \pi D + \alpha(\pi - q).$$

The solution depends on the relationship between the insurance price q and the probability π:

- If $\pi = q$, then $\pi - q = 0$ and expected wealth

  $$w - \pi D$$

  is constant in α. Expected wealth is the same for every α, so the decision maker is indifferent over all insurance levels.

- If $\pi < q$, then the slope $\pi - q < 0$, so expected wealth

  $$w - \pi D + \alpha(\pi - q)$$

  is strictly decreasing in α. Expected wealth is maximised by choosing no insurance, $\alpha^* = 0$.

If the individual is risk averse, and therefore u is strictly concave, then:

- With *fair insurance*, $q = \pi$, expected wealth does not depend on α, but full insurance, $\alpha = D$, makes wealth non-random, as it is $w - \pi D$ in both states. A strictly risk-averse individual strictly prefers this certainty and chooses full insurance. You are asked to verify this claim in Exercise 4.4.

- With $q > \pi$, expected wealth decreases in α, but insurance reduces risk. A strictly risk-averse decision maker may still choose a strictly positive but finite α^*, characterised by the first-order condition above.

Risky asset Consider an investor with expected utility preferences and initial wealth $w > 0$. There are two assets: a *safe asset* with sure return 1 per euro invested; a *risky asset* with random return z per euro invested, with distribution F and

$$\int z\, dF(z) > 1, \tag{4.3}$$

so that the risky asset has higher expected return than the safe asset.

Let α denote the amount invested in the risky asset and β the amount invested in the safe asset. The budget constraint is

$$\alpha + \beta = w, \qquad \alpha, \beta \geq 0.$$

For any realization z, final wealth is

$$\alpha z + \beta.$$

Using the budget constraint $\beta = w - \alpha$, we can rewrite wealth as

$$\alpha z + (w - \alpha) = w + \alpha(z - 1).$$

We now use the expectation notation $\mathbb{E}$, omitting the CDF F for simplicity. The investor's problem is thus

$$\max_{0 \leq \alpha \leq w} \mathbb{E}\big[u\big(w + \alpha(z - 1)\big)\big]. \tag{4.4}$$

Suppose the optimum is interior, $\alpha^* \in (0, w)$. Then the first-order condition for α^* is

$$\mathbb{E}\big[\, u'\big(w + \alpha^*(z - 1)\big)\, (z - 1) \,\big] = 0.$$

Notice that this condition resembles the one we obtained for insurance. In each case, the individual trades off marginal utility in different states weighted by the "per-unit payoff difference" in that state.

If the individual is risk neutral and therefore u is linear, then u' is constant and the condition reduces to

$$\mathbb{E}[z - 1] = 0.$$

Since $\mathbb{E}[z] > 1$ by Equation (4.3), $\mathbb{E}[z-1] > 0$, so the derivative of expected utility is positive for all α and the optimum is a corner: $\alpha^* = w$, all wealth is invested in the risky asset.

If instead u is strictly concave, then u' is decreasing, so high-z states, where wealth is high, are given less weight and low-z states are given more weight. The higher expected return of the risky asset is traded off against the disutility of risk, and this typically yields an interior solution $0 < \alpha^* < w$.

If u is twice differentiable and strictly concave, then the second derivative of expected utility with respect to α is

$$\mathbb{E}\big[\, u''\big(w + \alpha(z-1)\big)\,(z-1)^2 \,\big] < 0,$$

since $(z-1)^2 \geq 0$ and $u'' < 0$. Hence expected utility is strictly concave in α, so any solution to the first-order condition is the unique global maximizer.

Evaluating the derivative at $\alpha = 0$ gives

$$\mathbb{E}\big[\, u'(w)(z-1)\big] = u'(w)\,\mathbb{E}[z-1].$$

If $\mathbb{E}[z] > 1$, then $\mathbb{E}[z-1] > 0$, so the derivative at $\alpha = 0$ is positive and the individual strictly prefers to hold a positive amount of the risky asset.

From the condition

$$\mathbb{E}\big[\, u'\big(w + \alpha^*(z-1)\big)\,(z-1)\,\big] = 0,$$

one could argue that the optimal risky position α^* decreases if the individual becomes more risk averse. That is, u becomes more concave, so bad states are weighted more heavily through u'. The moral of the story is: if a risk is actuarially favourable, a risk-averse individual will invest in it, but the more risk averse she is, the less she will invest.

Things to read. Most of this lecture draws from Mas-Colell et al. (1995, ch. 6.D.), you can find an alternative treatment in Kreps (2013, ch. 6.3).

4.3 Exercises

Exercise 4.1. Prove one direction of Proposition 4.1: show that if F first-order stochastically dominates G in the sense of Definition 4.1, then $F(x) \leq G(x)$. (check Mas-Colell et al. (1995, p. 195) if you are stuck)

Exercise 4.2. There is another way of defining first-order stochastic dominance. Consider the following compound lottery. First, an outcome x is drawn according to a distribution F. Then, to the realisation x, an amount z, distributed according to G, is added. Show that such a compound lottery first-order stochastically dominates F. The reserve also hold: if F first-order stochastically dominates G, then F can be constructed as a compound lottery as above!

Solution to Exercise 4.2. Check Mas-Colell et al. (1995, Example 6.D.1, p. 196). □

Exercise 4.3. Prove one direction of Proposition 4.2, if G is a mean preserving spread of F, then F second-order stochastically dominates G. (check Mas-Colell et al. (1995, p. 197) if you are stuck)

Exercise 4.4. Consider the insurance problem in Equation (4.2) with fair insurance, $q = \pi$. Show that full insurance, $\alpha = D$, is optimal for a risk-averse individual.

Solution to Exercise 4.4. Check Mas-Colell et al. (1995, Example 6.C.1, p. 187-188). □

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Lecture 5

States and subjective expected utility

5.1 State space representation

Until now we studied a framework of uncertainty in which the underlying state generating the probability of outcomes was not modelled explicitly, as discussed in Remark 1.1. There are two advantages of modelling uncerlying states of the world explicitly. The first is that the individual might care about the state *per se*. Consider the following example.

Example 5.1. The birthday of your child is coming up. The problem is that you do not know whether it will rain or be sunny that day. You are a sophisticated parent who offers him monetary bets on the climate whose payoffs he can spend on his birthday party. If it is sunny, he will have a great time playing outside with his friends, while if it rains he will be obliged to organise something indoors. Therefore, he may enjoy each euro spent on his birthday more when it is sunny than when it is raining: his preferences over money depend on the weather.[1] ■

The first advantage of modelling states explicitly is that it allows us to capture preferences that depend on the state of the world, as in Example 5.1. There is a second advantage of modelling states explicitly, but it is easier to explain after we introduce the framework. As in Lecture 1, there is a finite set of outcomes X, in Example 5.1 these are the amounts of money the child could get. Moreover, there is a finite set of mutually exclusive states of the world S. In the words of Arrow (1971, p. 45) each state is "a description of the world so complete that, if true and known, the consequences of every action would be known". In Example 5.1, these are the weather conditions, rain or sun. The individual chooses an **act**, which is a function from states to outcomes $f : S \to X$. Act f in state s leads to the outcome f_s. In Example 5.1, an act is a state-contingent bet. If it rains, the child gets f_{rain} euros, while if it is sunny he gets f_{sun} euros. Acts are referred to as **Savage acts**, after Savage (1972),[2] who introduced the framework and derived subjective expected utility in Definition 5.3 below.

We can now discuss the second advantage of modelling states explicitly. In Lecture 1, the individual chooses among lotteries, probability distributions over outcomes. From individual preferences over lotteries, we can infer his Bernoulli utility over outcomes u, and various properties it might have, such as risk aversion. However, the probability of realisation of outcomes is *given*. Most of the time, it is not clear what the probability of

[1] The example is inspired by Tsakas (2025)

[2] The first edition was published in 1954.

an outcome is, and the individual might have her own beliefs about these probabilities. Notice that, in the current framework, we have not introduced any probability yet. The idea is that we want to *infer* both the individual's utility over outcomes u and her beliefs about the likelihood of states p from her preferences over acts. We proceed as we did in Lecture 1, by studying preferences over acts that have a functional representation of interest.

You should notice that, in this setting, there is no natural mixing operation comparable to the one we had for lotteries in Lecture 1.

5.2 Subjective expected utility

We now study preferences over acts, that is, if $f \succsim f'$ we say the individual weakly prefers act f to act f'. The set of all acts is denoted by X^S, i.e., the set of all functions from S to X. The definition of a utility function representing preferences is analogous to Definition 1.1.

Definition 5.1. *A utility function $U \colon X^S \to \mathbb{R}$* ***represents*** *the preference relation $\succsim$ over acts if, for all acts f, f',*

$$f \succsim f' \iff U(f) \geq U(f').$$

Under suitable conditions on preferences $\succsim$, we can represent them through a form of expected utility paralleling the one we considered until now.

Definition 5.2. *Preferences $\succsim$ over acts have a* ***state-dependent subjective expected utility*** *representation if there exists a probability distribution over states $p \in \Delta(S)$ and, for each state s, a utility function over outcomes $u_s : X \to \mathbb{R}$ such that, for all acts f,*

$$U(f) = \sum_s p(s) u_s(f_s). \tag{5.1}$$

Let us discuss the interpretation of Definition 5.2. The individual has *subjective* beliefs about the likelihood of states, represented by the probability distribution p. Moreover, she has preferences over outcomes that depend on the state of the world, represented by the state-dependent utility functions u_s. The individual evaluates each act f by computing its expected utility according to his subjective beliefs p, as represented by Equation (5.1), and prefers acts with higher expected utility.

If you think Equation (5.1) is the same as objective expected utility from Lecture 1, think twice. First, we could not define a state-dependent utility u_s, because there were no states. But second, and more importantly, in objective expected utility the probabilities were *given*, while here they are *subjective*, that is, they represent the individual's beliefs about the likelihood of states. We can infer beliefs from preferences. In other words, if you compare two individuals with distinct preferences over acts, they might have different beliefs about the likelihood of states, even if they have the same utility over outcomes.

A question you might ask yourself is to what extent preferences u_s and beliefs p are unique, as we did for objective expected utility in Lecture 1. The answer to this question poses problems for the interpretation of sate-dependent subjective expected utility we gave above. Consider the following transformation of u:

$$\widetilde{u}_s = \alpha_s + \beta_s \frac{p(s)}{\widetilde{p}(s)} u_s,$$

for $\alpha_s \in \mathbb{R}$ and $\beta_s > 0$, and $\widetilde{p} \in \Delta(S)$. We can then compute:

$$\begin{aligned}
\widetilde{U}(f) &= \sum_s \widetilde{p}(s)\, \widetilde{u}_s(f(s)) \\
&= \sum_s \widetilde{p}(s) \left(\alpha_s + \beta_s \frac{p(s)}{\widetilde{p}(s)} u_s(f(s)) \right) \\
&= \sum_s \widetilde{p}(s) \alpha_s + \sum_s \beta_s p(s)\, u_s(f(s)) \\
&= \alpha + U(f).
\end{aligned}$$

Therefore, $\widetilde{U}(f)$ represents the same preferences as $U(f)$. We are not able to identify beliefs and preferences uniquely. The statement "an individual prefers act f to act f' because she believes state s is very likely and likes outcome x a lot in that state" is not well defined, as we can change beliefs and preferences in a way that leaves preferences over acts unchanged.[3] (Kreps, 1988, p. 36) suggests that it would be more appropriate to write Equation (5.1) as

$$U(f) = \sum_s v_s(f_s),$$

that is, state-dependent subjective expected utility is just additive separability across states.

We can solve this identification problem by imposing that preferences over outcomes do not depend on the state of the world, that is, $u_s = u$ for all states s. In this case, we obtain the following definition.

Definition 5.3. *Preferences $\succsim$ over acts have a **subjective expected utility** representation if there exists a probability distribution over states $p \in \Delta(S)$ and a utility function over outcomes $u : X \to \mathbb{R}$ such that, for all acts f,*

$$U(f) = \sum_s p(s) u(f_s). \tag{5.2}$$

[3]Unfortunately, this identification problem is often put under the rug, leading to sloppy interpretations of the role of beliefs.

In Definition 5.3, contrary to Definition 5.2, individual preferences over outcomes do not depend on the state. Such model has stronger uniqueness properties: if (p, u) and $(\widetilde{p}, \widetilde{u})$ both represent preferences through Equation (5.2), then there exist $\alpha \in \mathbb{R}$ and $\beta > 0$ such that $\widetilde{u} = \alpha + \beta u$ and $\widetilde{p} = p$. Therefore, beliefs p are uniquely identified, while utility u is identified up to positive affine transformations, as in objective expected utility. In this case we can interpret the probability p as the individual's subjective beliefs about the likelihood of states.

What assumptions over preferences over acts are equivalent to the existence of a subjective expected utility representation? The answer is given by Savage's Theorem (Savage, 1972). Unfortunately, such axiomatic analysis is beyond the scope of this lecture. However, we will focus on the main axiom that allows us to obtain subjective expected utility, the **Sure-Thing Principle**. To state it, we need some notation.

Recall that an *event* is a subset of states $E \subseteq S$. For each event E, we write E^c for the complement of E in S, that is, the set of states in S that are not in E. For any two acts f, g and any event $E \subseteq S$ define an act $f_E g$ such that

$$f_E g(s) = \begin{cases} f(s) & \text{if } s \in E \\ g(s) & \text{if } s \in E^c \end{cases}$$

That is, act $f_E g$ agrees with act f on states in event E, and with act g on states outside event E.

Axiom 5.1. ***(Sure-thing principle)*** *For all acts f, g, f', g' and event E,*

$$f_E g \succsim f'_E g \text{ if and only if } f_E g' \succsim f'_E g'.$$

In words, the Sure-thing principle says the following: if the individual prefers f to f' on states in event E, then what happens outside E should not matter. The ranking between f and f' on E should not be reversed by changing g to g' outside E. The Sure-thing principle is the key axiom of Subjective Expected Utility. Savage (1972) showed that the Sure-thing principle, together with other axioms, is equivalent to the existence of a subjective expected utility representation as in Definition 5.3, an axiomatic analysis paralleling the one we developed for objective expected utility in Theorem 2.1.

However, as for Independence, we have empirical evidence that individuals' choices sometimes violate the Sure-thing principle. The most famous example is the Ellsberg paradox, by Ellsberg (1961). Consider the following tought experiment. An urn contains 90 balls, of which 30 are red, while the remaining 60 are either black or yellow, in an unknown proportion. The state space in this example comprises therefore the colors of the balls: $S = \{R, B, Y\}$. You are offered to bet on the colour of a randomly drawn ball from the urn. You can choose between the following acts:

- f: You win 1 euro if the ball is red, and nothing otherwise.

- g: You win 1 euro if the ball is black, and nothing otherwise.
- f': You win 1 euro if the ball is red or yellow, and nothing otherwise.
- g': You win 1 euro if the ball is black or yellow, and nothing otherwise.

These acts can be summarised in the following table:

	R	B	Y
f	1	0	0
g	0	1	0
f'	1	0	1
g'	0	1	1

Many people prefer f to g, and g' to f'. However, such preferences violate the Sure-thing principle. Let $E = \{R, B\}$ and consider two acts h^0 and h^1 such that

$$h^0(s) = 0 \quad \text{and} \quad h^1(s) = 1 \quad \text{for all } s.$$

Then we can rewrite the four acts in the table as

$$f = f_E h^0, \qquad g = g_E h^0, \qquad f' = f_E h^1, \qquad g' = g_E h^1.$$

Apply the Sure-thing principle with the acts f, g, h^0, h^1 and event E. The axiom says that

$$f_E h^0 \succsim g_E h^0 \iff f_E h^1 \succsim g_E h^1,$$

that is,

$$f \succsim g \iff f' \succsim g'.$$

Hence it is impossible to have $f \succ g$ and $g' \succ f'$ without violating the Sure-thing principle.

One explanation for this behaviour is that people dislike **ambiguity**, that is, situations in which the likelihood of states is unknown. In the Ellsberg paradox there are 30 red balls and 60 balls that are either blue or yellow, in unknown proportions. Thus, the probability of drawing a red ball is known, while the probability of drawing a blue ball is unknown. In the first pair, act f pays 1 only if R occurs, so it yields a known probability of winning of $\frac{1}{3}$. Act g pays 1 only if B occurs, so its probability of winning depends on the unknown fraction of blue balls in the urn. Many people therefore prefer the bet with known probability f to the bet with unknown probability g.

In the second pair, act f' pays 1 if either R or Y occurs. Since the probability of B is unknown, the probability of $R \cup Y$ is also unknown: it could be anywhere between $\frac{1}{3}$ and 1. By contrast, g' pays 1 if either B or Y occurs, and the probability of $B \cup Y$ is known to be $\frac{2}{3}$. So in this case people tend to prefer g', a bet with known probability $\frac{2}{3}$, to f', a bet with unknown probability.

A plethora of theories of choice under uncertainty have been proposed to account for such violations of the Sure-thing principle. One of the most influential branches attempts to explain the Ellsberg paradox with **ambiguity aversion**, therefore introducing a notion of preferences exhibiting such a property. The seminal paper is Schmeidler (1989).

Any ideas on how the Sure-thing principle could be relaxed to account for the Ellsberg paradox?

Things to read. For a textbook treatment of the content of this lecture, see Kreps (1988, pp. 33-38) or Fishburn (1970, ch. 12). If you are interested in more details, read Kreps (1988, Chs. 8-9) or Fishburn (1970, ch. 14). By the way, Kreps (1988, p. 127) defines Savage (1972)'s theory nothing less than "the crowning achievement of single-person decision theory". At this point of the class, you might be interested in reading Gilboa (2009) for an overview of our current understanding of decision-making under uncertainty.

5.3 Exercises

Exercise 5.1. Show that the subjective expected utility representation in Definition 5.3 satisfies the Sure-thing principle.

Solution to Exercise 5.1. Let f, g, f', g' be acts and $E \subseteq S$ an event. Assume that $f_E g \succsim f'_E g$. By Definition 5.3, this means that

$$\sum_{s\in E} p(s)u(f(s)) + \sum_{s\in E^c} p(s)u(g(s)) \geq \sum_{s\in E} p(s)u(f'(s)) + \sum_{s\in E^c} p(s)u(g(s)).$$

Subtracting the common term $\sum_{s\in E^c} p(s)u(g(s))$ from both sides, we obtain

$$\sum_{s\in E} p(s)u(f(s)) \geq \sum_{s\in E} p(s)u(f'(s)).$$

By adding the common term $\sum_{s\in E^c} p(s)u(g'(s))$ to both sides, we get

$$\sum_{s\in E} p(s)u(f(s)) + \sum_{s\in E^c} p(s)u(g'(s)) \geq \sum_{s\in E} p(s)u(f'(s)) + \sum_{s\in E^c} p(s)u(g'(s)),$$

that is, $f_E g' \succsim f'_E g'$. The reverse implication is analogous. $\square$

Exercise 5.2. Can you find a parallel between the Sure-thing principle and the Independence from Lecture 2? Think about compound lotteries and acts that agree on all states except one.

Solution to Exercise 5.2. Consider two lotteries p, p' and a third lottery q. Consider a probability $\alpha \in (0,1)$ and the compound lotteries $\alpha p + (1-\alpha)q$ and $\alpha p' + (1-\alpha)q$. Independence says that

$$p \succsim p' \iff \alpha p + (1-\alpha)q \succsim \alpha p' + (1-\alpha)q.$$

Now consider two acts f, f' and a third act g. Consider an event $E \subseteq S$ and the acts $f_E g$ and $f'_E g$. The Sure-thing principle says that

$$f \succsim f' \iff f_E g \succsim f'_E g.$$

Both axioms say that the ranking between two objects, lotteries or acts, should not be affected by mixing them with a third object, lottery or act, that is the same in both cases. □

Exercise 5.3. There is a second important model of uncertainty using a state space, by Anscombe & Aumann (1963).[4] In this model, the individual chooses acts mapping state of the world to lotteries, rather than outcomes. That is, each act is a function $f : S \to \Delta(X)$. Write down a subjective expected utility representation for this model. What do you think the advantages of this model are? (Think about the remark about mixing in the text.)

Solution to Exercise 5.3. In this model, preferences $\succsim$ over acts have a **subjective expected utility** representation if there exists a probability distribution over states $p \in \Delta(S)$ and a utility function over outcomes $u : X \to \mathbb{R}$ such that, for all acts f,

$$U(f) = \sum_s p(s) \sum_{x \in X} f[s](x)u(x).$$

The advantage of this model is that we can define mixing of acts naturally. Given two acts f, g and a probability $\alpha \in (0,1)$, we can define the mixed act $h = \alpha f + (1-\alpha)g$ such that

$$h(s) = \alpha f(s) + (1-\alpha)g(s),$$

that is, in each state the mixed act yields the mixed lottery. This allows us to define an independence axiom for acts analogous to the one we had for lotteries in Lecture 2. □

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[4]What is mostly used today is the version in Fishburn (1970).

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Lecture 6

Introduction to exchange economies

6.1 A primer on consumer choice

Before we turn to general equilibrium, we need to review a few basic concepts about consumer choice. The setting is intuitive. There is a single individual i who consumes bundles of ℓ different goods. The quantity of each good is represented by a real number. The consumption space is therefore $\mathbb{R}^\ell_+$. A generic consumption bundle for individual i is $x_i = (x_i^1, \ldots, x_i^\ell)$. We use subscripts for individuals and superscripts for goods. For example, if $\ell = 2$, a consumption bundle could be $x_i = (3, 5)$, meaning 3 units of good 1 and 5 units of good 2, as represented in Figure 6.1. We index consumption bundles by i because later we will consider multiple individuals, each with their own consumption bundle.

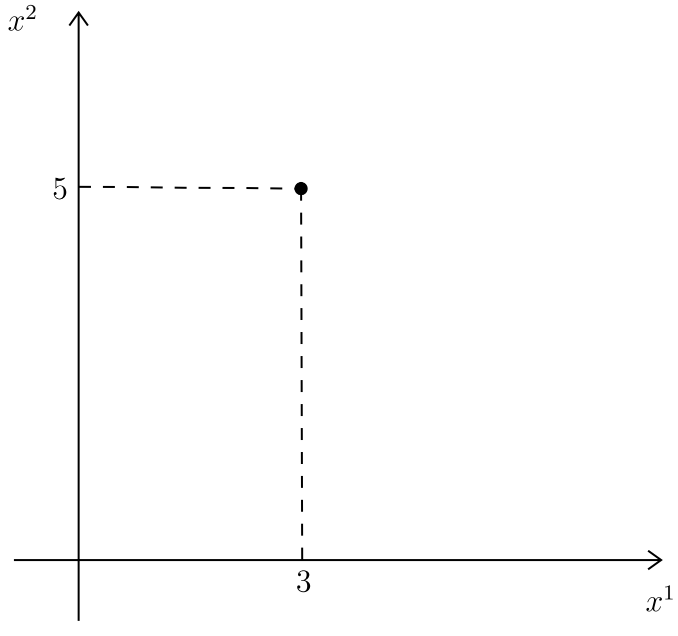

Figure 6.1: A consumption bundle $x_i = (3, 5)$ in a two-good consumption space.

Now suppose there is a vector of prices $p = (p^1, \ldots, p^\ell)$, where p^k is the price of good k. Also assume that the individual has monetary wealth $w_i \in \mathbb{R}_+$. He can therefore consume any bundle x_i such that total expenditure does not exceed w_i, that is, such that $p \cdot x_i \leq w_i$.[1] The set of all consumption bundles that satisfy this condition is called the **budget set**, and is denoted by

[1]The operation $\cdot$ denotes the product $p^1 x_i^1 + \cdots + p^\ell x_i^\ell$.

$$B(p, w_i) = \{x_i \in \mathbb{R}^\ell_+ \mid p \cdot x_i \leq w_i\}.$$

The budget set is illustrated in Figure 6.2 in the two-good case. The budget line is the boundary of the budget set: it consists of all consumption bundles x_i such that $p \cdot x_i = w_i$. If the individual consumes only good 1, then setting $x_i^2 = 0$ yields $x_i^1 = \frac{w_i}{p^1}$. Similarly, if he consumes only good 2, then setting $x_i^1 = 0$ yields $x_i^2 = \frac{w_i}{p^2}$. These two points are the intercepts of the budget line. The slope of the budget line is the relative price

$$w_i = p^1 x_i^1 + p^2 x_i^2 \implies x_i^2 = \frac{w_i}{p^2} - \frac{p^1}{p^2} x_i^1.$$

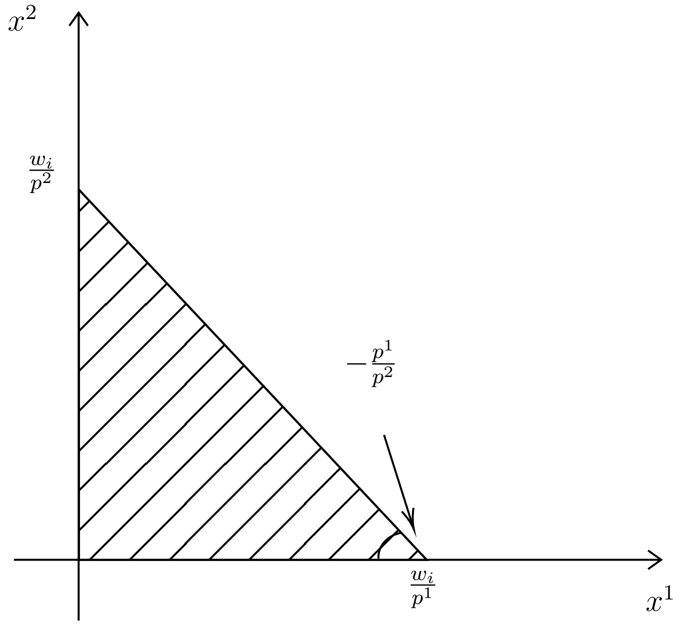

Figure 6.2: A budget set $B(p, w_i)$ in a two-good consumption space.

The individual has preferences over consumption bundles. In previous lectures we studied choice under uncertainty, where the outcome of a choice is to some extent random. Here there is no uncertainty: the individual has preferences $\succsim_i$ over the consumption space $\mathbb{R}^\ell_+$. As we did for the simplex, we can visualise these preferences by drawing indifference curves in the consumption space, as in Figure 6.3. All bundles on the same indifference curve are equally good for the individual.

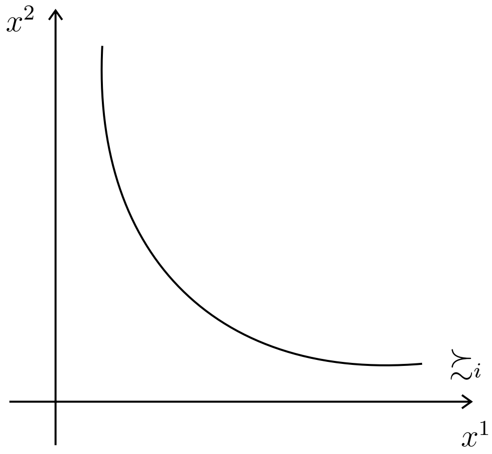

Figure 6.3: An indifference curve in the consumption space $\mathbb{R}^2_+$.

The individual would like to choose the most preferred bundle among those he can afford. Affordability is determined by the budget, and therefore by prices and wealth. Any most preferred affordable bundle is what the individual "demands" at the given prices and wealth. The **Walrasian demand** of an individual with preferences $\succsim_i$, at prices p and wealth w_i, is

$$D_i(p, w_i) = \{x_i \in B(p, w_i) \mid x_i \succsim_i x_i' \text{ for all } x_i' \in B(p, w_i)\}.$$

In other words, a bundle x_i belongs to $D_i(p, w_i)$ if it is affordable and at least as good as every other affordable bundle. In general, Walrasian demand is a set: it may happen that several affordable bundles tie for being best. We can visualise Walrasian demand graphically. For example, for the preferences represented in Figure 6.3, and assuming preferences are increasing in each good, Walrasian demand is a single point, as shown in Figure 6.4.

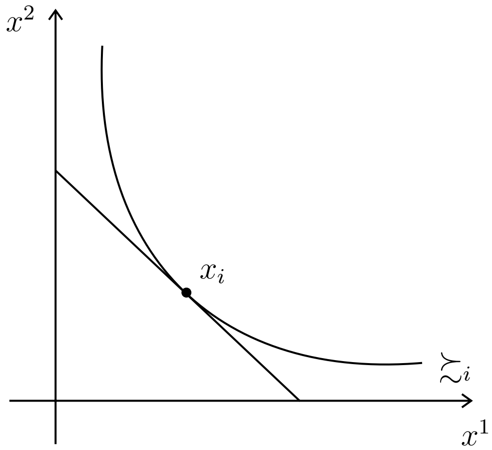

Figure 6.4: Walrasian demand for preferences $\succsim_i$.

What kind of indifference curves should $\succsim_i$ have for the Walrasian demand to contain more than one element? Can you construct an example in the graph?

In the next section, we build on these basics to consider the choices of two individuals simultaneously.

6.2 Illustrative example of exchange economy

We now move from individual choice, which we studied in the previous lectures, to *collective* choice. In general equilibrium theory, we generalise concepts from individual consumer choice to an economy with multiple individuals. In this section we start from a simple example with two individuals and two goods, which we will later generalise.

Suppose there are two individuals 1 and 2 and two goods. Each individual has the consumption space $\mathbb{R}^2_+$, and a consumption bundle $x_i = (x_i^1, x_i^2)$. We can represent the consumption space of individual 1, together with his indifference curves, as we did in Figure 6.3. For individual 2, we can do the same, but suppose we draw his consumption space upside down, as in Figure 6.5 (bear with me).

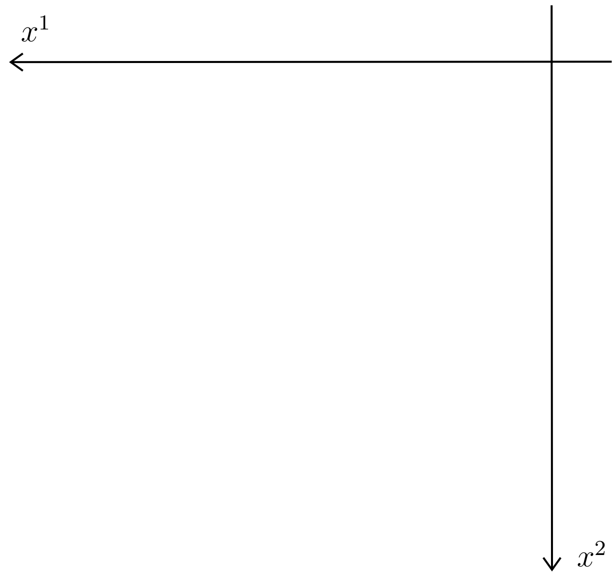

Figure 6.5: Consumption space of individual 2 upside down.

Now we can combine the two consumption spaces in a single graph, called the **Edgeworth box**, as in Figure 6.6. The total width of the box is the total amount of good 1 available in the economy as a whole, and the total height is the total amount of good 2 available. The two origins O_1 and O_2 are at the bottom left and top right corners of the box, respectively. Each point in the box represents an **allocation** of goods between the two individuals. For example, the point x represents the allocation in which individual 1 consumes (x_1^1, x_1^2) and individual 2 consumes (x_2^1, x_2^2). The consumption of each individual i is read by viewing the box from the perspective of origin O_i.

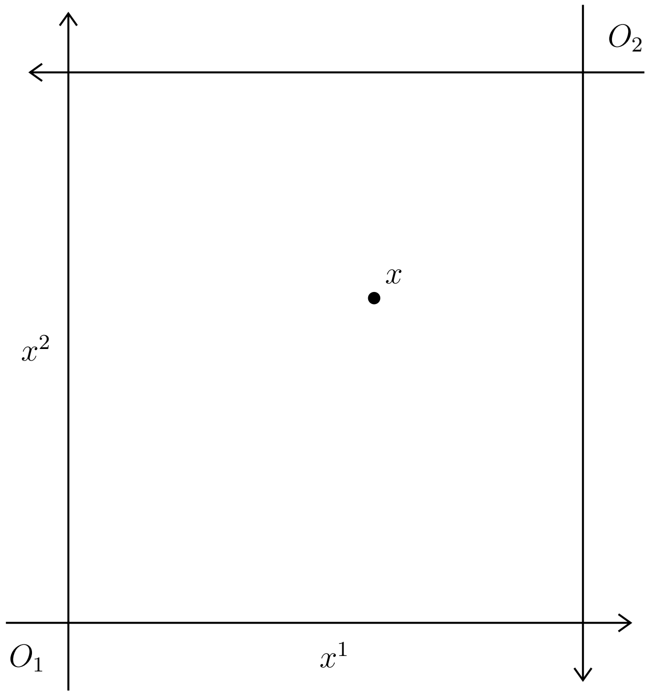

Figure 6.6: Edgeworth box representing the consumption spaces of individuals 1 and 2.

We can also represent the indifference curves of both individuals through x in the Edgeworth box, as in Figure 6.7. The indifference curve of individual i is indicated with $\succsim_i$.

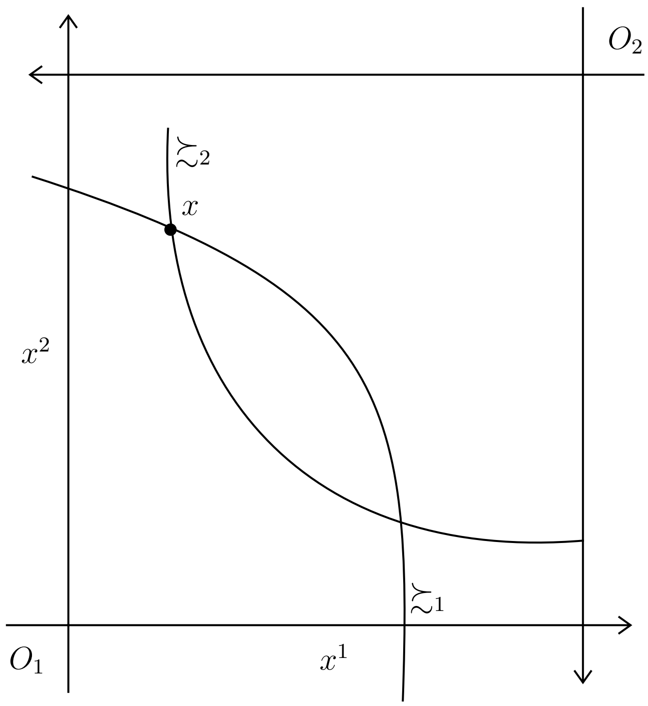

Figure 6.7: Edgeworth box representing the consumption spaces of individuals 1 and 2.

We assume that preferences are increasing in each good. Therefore, both individuals would like to move away from the origin of their own consumption space. For example, individual 1 prefers being on the dotted indifference curve rather than on the solid one in Figure 6.7.

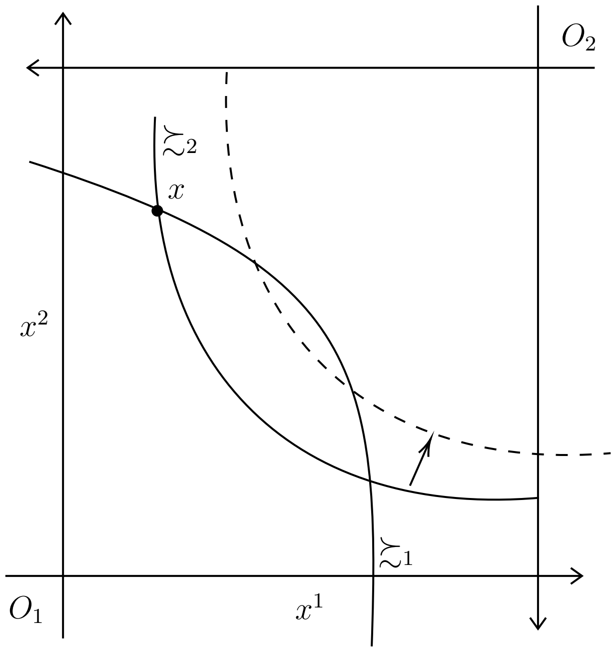

Figure 6.8: Individual 1 would prefer to move on the dotted indifference curve.

Now suppose that each individual has an initial endowment of goods, that is, a bundle of goods that he initially owns. The endowment of individual i is denoted by $e_i = (e_i^1, e_i^2)$. The initial endowments determine the initial allocation in the Edgeworth box. The total endowment in the economy is $\bar{e} = e_1 + e_2$, and this determines the size of the Edgeworth box. In principle, individuals can trade their endowments to reach preferred allocations.

Now suppose there are prices in the economy, given by the vector $p = (p^1, p^2)$, one for each good. Given prices and endowments, we can compute the initial wealth of each individual. In individual consumer choice, wealth was given as a monetary amount w_i, whereas here it is the value of the endowment at the given prices. The wealth of individual i is therefore $w_i = p \cdot e_i = p^1 e_i^1 + p^2 e_i^2$. Hence, the budget set of individual i consists of all consumption bundles x_i such that $p \cdot x_i \leq p \cdot e_i$. So the budget set is[2]

$$B(p, e_i) = \{x_i \in \mathbb{R}_+^2 \mid p \cdot x_i \leq p \cdot e_i\}.$$

We can draw the budget line in the Edgeworth box, as in Figure 6.9. First, it passes through the endowment point, since the individual can always afford to consume his initial endowment. Second, the slope of the budget line is given by the relative price, as we saw in individual consumer choice.

[2] There is a slight abuse of notation: to be fully consistent with the earlier definition, I should write $B(p, p \cdot e_i)$.

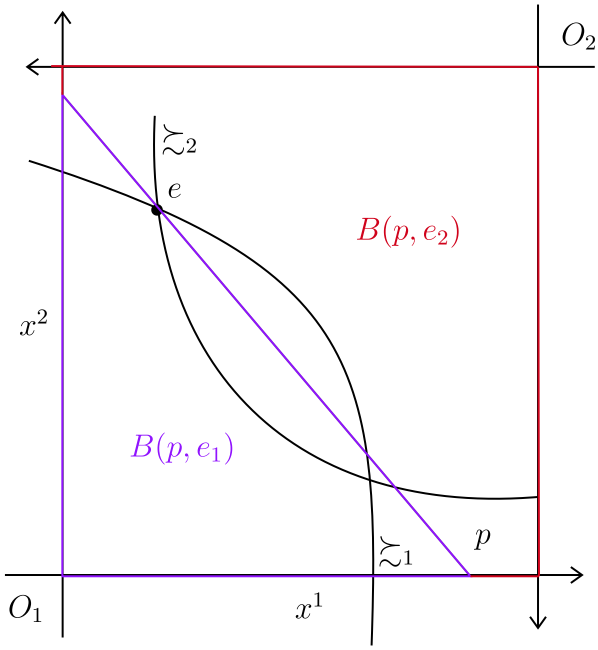

Figure 6.9: Budget line and budget sets in the Edgeworth box.

Individuals may then choose their favourite consumption bundle in their budget set according to their preferences, that is, a bundle in their Walrasian demand set. When both individuals choose a bundle from their Walrasian demand set at the same prices, we can ask whether total demand is equal to the total endowment in the economy. If this is the case, we have found a **Walrasian equilibrium**, as illustrated in Figure 6.10.

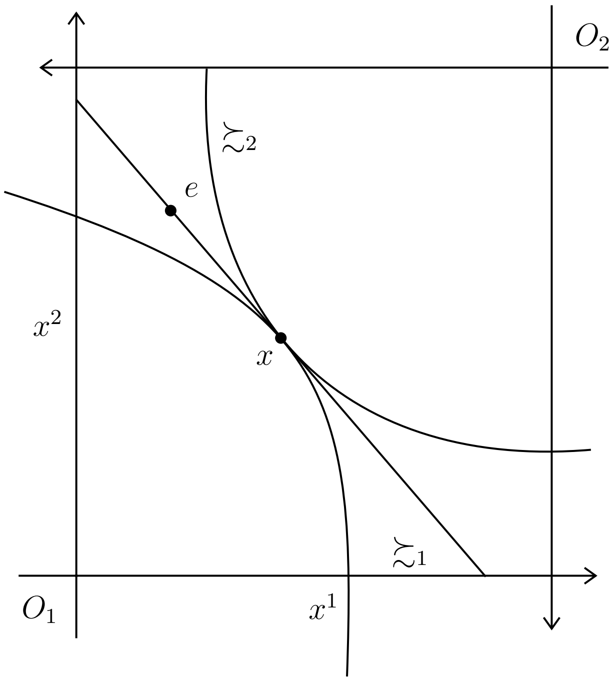

Figure 6.10: Walrasian equilibrium x in the Edgeworth box.

In the rest of the lectures, we mainly focus on the properties of Walrasian equilibria. One might wonder whether Walrasian equilibria induce allocations that are desirable in some sense. Let us consider a possible criterion of desirability. Take an allocation x in the Edgeworth box. Is there any other allocation x' such that both individuals weakly prefer x' to x, and at least one of them strictly prefers it? If such an allocation x' exists, then we say that x' is a **Pareto improvement** over x, since at least one individual is strictly better off and no individual is worse off. If no such allocation x' exists, then we say that x is **Pareto optimal**. As an example, the endowment allocation e in Figure 6.10 is **not** Pareto optimal. Pareto optimality is arguably a minimal requirement for an allocation to be considered desirable. The first theorem of welfare economics states that, under some assumptions, any Walrasian equilibrium allocation is Pareto optimal, and therefore that Walrasian equilibria satisfy this minimal requirement of desirability.[3] In fact, the Walrasian equilibrium allocation x in Figure 6.10 is Pareto optimal.

However, Pareto optimality is sometimes too weak. For example, the corners of the Edgeworth box are Pareto optimal, but one might argue that they are not desirable allocations. Pareto optimality is a requirement of *efficiency*. No resource is wasted in achieving preference satisfaction. An efficiency requirement might be complemented by a *fairness* requirement. Luckily, there is an extensive literature on fairness in social choice theory, deeply informed by philosophical work.[4]

A notion of fairness that has been widely studied is that of **envy-freeness**.[5] An allocation x is envy-free if no individual prefers another individual's bundle to his own. In other words, individual i does not envy individual j if $x_i \succsim_i x_j$. An allocation is envy-free if, for every pair $i \neq j$, individual i does not envy individual j. Envy-freeness is related to equality of opportunities, for reasons that we clarify in the next lectures. For now, just note that the corner allocations in the Edgeworth box are not envy-free under our assumptions.

We know from the first theorem of welfare economics that Walrasian equilibria are Pareto optimal. But what if we want to select particular Pareto efficient allocations, for example those that are also envy-free? The second theorem of welfare economics states that, under suitable assumptions, any Pareto optimal allocation can be supported by a Walrasian equilibrium, following a redistribution of initial endowments. Therefore, if there exists an envy-free Pareto optimal allocation, then there exists a Walrasian equilibrium that supports it. It looks as though Walrasian equilibria can deliver desirable allocations, after a suitable redistribution of initial endowments.[6]

[3] However, some people reject Pareto optimality. One reason is that it is defined entirely in terms of preferences, while we might want to evaluate allocations using other considerations.

[4] If you are interested, check Moulin (1988), Young (1994), Roemer (1996), Moulin (2004), Fleurbaey (2008), Fleurbaey & Maniquet (2011), Thomson (2011). Gabriel Carroll has a syllabus for an interesting class here, if you want to explore further.

[5] Apparently it has been introduced by Jan Tinbergen (Heilmann & Wintein, 2021).

[6] Aviad Heifetz suggested this motivation to me for discussing the second theorem of welfare economics.

Things to read. It might be useful for you to review (or study, if you have never encountered these topics before) Hildenbrand & Kirman (1976, pp. 51–70, 76–84). If you want (and you "should want") to go further, study Mas-Colell et al. (1995, pp. 17–23, 40–56). There is a worked-out example of a simple exchange economy in Mas-Colell et al. (1995, pp. 515–525). You can play with this online tool to visualise the Edgeworth box and envy-free allocations under different preferences and endowments. It was shared with me by Aviad Heifetz.

6.3 Exercises

Exercise 6.1. Prove that Walrasian demand satisfies the following property: for any prices p, wealth w_i, and any scalar $\alpha > 0$,

$$D_i(\alpha p, \alpha w_i) = D_i(p, w_i).$$

This property is called **homogeneity of degree zero** of Walrasian demand.

Solution to Exercise 6.1. Let $x_i \in D_i(p, w_i)$. Now consider the budget set $B(\alpha p, \alpha w_i)$. A bundle z_i belongs to this budget set if and only if

$$\alpha p \cdot z_i \leq \alpha w_i.$$

Dividing both sides by $\alpha > 0$, we get

$$p \cdot z_i \leq w_i,$$

which means that $z_i \in B(p, w_i)$. Therefore, the budget sets are equal:

$$B(\alpha p, \alpha w_i) = B(p, w_i).$$

Since the budget sets are the same, the maximisation problem faced by the individual is unchanged. Therefore, the Walrasian demand sets must also be equal:

$$D_i(\alpha p, \alpha w_i) = D_i(p, w_i).$$

□

Exercise 6.2. Assume that preferences $\succsim_i$ are increasing in each good. Prove that, for any strictly positive prices p and wealth w_i, the Walrasian demand $D_i(p, w_i)$ contains only bundles x_i that satisfy $p \cdot x_i = w_i$. This property is referred to as **Walras' law**.

Solution to Exercise 6.2. Let $x_i \in D_i(p, w_i)$. Suppose, for the sake of contradiction, that $p \cdot x_i < w_i$. Since preferences are increasing in each good, there exists a bundle z_i such that $z_i \succ_i x_i$ and $p \cdot z_i \leq w_i$. This means that $z_i \in B(p, w_i)$ and $z_i \succ_i x_i$, which contradicts the fact that x_i is in the Walrasian demand set. Therefore, it must be the case that $p \cdot x_i = w_i$ for all $x_i \in D_i(p, w_i)$. □

Exercise 6.3. An individual has to choose consumption today c^1 and consumption tomorrow c^2. He has an initial endowment of wealth w^1 today and w^2 tomorrow. He can save or borrow at an interest rate r. What is the budget constraint of the individual?

Solution to Exercise 6.3. The individual can transfer wealth between today and tomorrow by saving or borrowing. If he saves an amount s today, he will have $(1+r)s$ available for consumption tomorrow. Conversely, if he borrows an amount b today, he will have to repay $(1+r)b$ tomorrow.

The budget constraint can be expressed as follows:

$$c^1 + \frac{c^2}{1+r} \leq w^1 + \frac{w^2}{1+r}.$$

The present value of consumption today and tomorrow cannot exceed the present value of the initial endowment. □

Exercise 6.4. An individual has to choose hours of work h and hours of leisure l. He has a total of T hours available, so that $h + l = T$. He earns a wage w per hour worked. What is the budget constraint of the individual in terms of consumption c and leisure l?

Solution to Exercise 6.4. The individual has a total of T hours available, which he can allocate between work h and leisure l. The individual's income from work is given by the wage rate w multiplied by the hours worked h:

$$\text{Income} = w \cdot h = w \cdot (T - l).$$

Assuming that the individual spends all his income on consumption c, the budget constraint can be expressed as:

$$c = w \cdot (T - l).$$

Rearranging this, we can express the budget constraint in terms of consumption c and leisure l:

$$c + w \cdot l = w \cdot T.$$

The total expenditure on consumption and the value of leisure cannot exceed the total income from working all available hours. □

Exercise 6.5. Assume that two individuals are in a situation like the one in Figure 6.8. Draw in the Edgeworth box the set of allocations that constitute an improvement over x for both individuals. In principle, they could trade to reach these allocations, without any outside intervention or money. Assume they bargain to reach one of these allocations. Which allocation would you expect them **not** to reach? Are these allocations fair?

Exercise 6.6. There is a simple graphical test to check whether an allocation in the Edgeworth box is envy-free. Look at Figure 6.11. Allocation x is **not** envy-free. The centre of the box is the point where each individual gets half of the total endowment. Allocation x' is the reflection of x with respect to the centre of the box. Why does this imply that allocation x is not envy-free? Can you understand the test? Draw an allocation that is envy-free and check that the graphical test works. You can use any indifference curves you like.

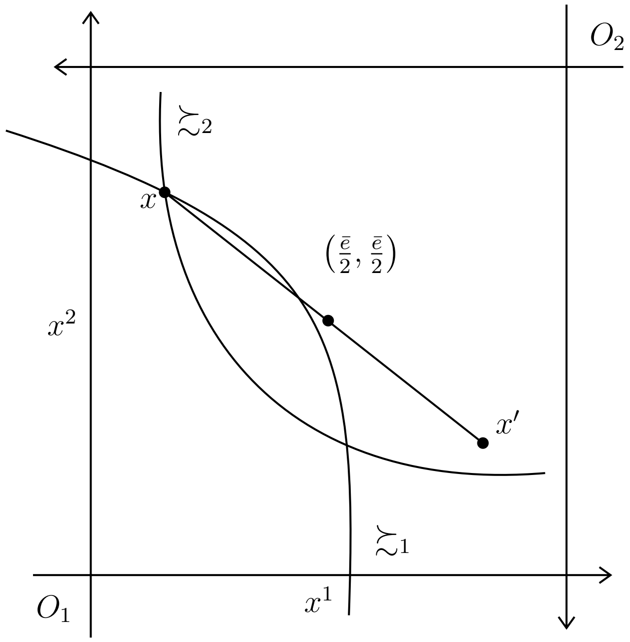

Figure 6.11: Graphical test for envy-freeness.

Solution to Exercise 6.6. Check Thomson (2011, Figure 21.1, p. 403). □

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Lecture 7

General equilibrium theory

7.1 Exchange economies

Primitives and definitions. We now generalise the example in Section 6.2. There is a set of n individuals $I = \{1, \ldots, n\}$ and the consumption space $\mathbb{R}^\ell_+$. Each individual has an endowment $e_i \in \mathbb{R}^\ell_+$, where $e_i = (e_i^1, \ldots, e_i^\ell)$. The total endowment in the economy is

$$\sum_i e_i = \bar{e}.$$

Each individual has preferences $\succsim_i$ over $\mathbb{R}^\ell_+$. A generic consumption bundle for individual i is $x_i = (x_i^1, \ldots, x_i^\ell)$. We always assume that preferences are complete and transitive for each i.

An **economy** is a profile $E = (\succsim_i, e_i)_{i \in I}$. An allocation x is **feasible** for E if

$$\sum_i x_i \leq \bar{e}.$$

Feasible allocations are elements of $\mathbb{R}^{\ell n}_+$. A price vector is $p = (p^1, \ldots, p^\ell) \in \mathbb{R}^\ell_+$, which assigns a price to each good. For each price vector, we define the budget set of an individual with endowment e_i.

Definition 7.1. *Given endowment e_i and prices p, the budget set is*

$$B(p, e_i) = \left\{ x_i \in \mathbb{R}^\ell_+ \;\middle|\; p \cdot x_i \leq p \cdot e_i \right\}.$$

Note that the budget set is always convex and, when all prices are strictly positive, it is also compact. We will often assume prices are strictly positive. Given prices p, endowment e_i, and preference relation $\succsim_i$, we define the Walrasian demand.

Definition 7.2. *The* ***Walrasian demand*** *of i, given endowment e_i and prices p, is*

$$D_i(p, e_i) = \{ x_i \in B(p, e_i) \mid x_i \succsim_i x_i' \text{ for all } x_i' \in B(p, e_i) \}.$$

The Walrasian demand is the set of most preferred bundles in the budget set.

Given a preference relation $\succsim_i$, we define the **upper and lower contour sets** at bundle x_i:

$$U_i(x_i) := \{ x_i' \in \mathbb{R}^\ell_+ \mid x_i' \succsim_i x_i \} \quad \text{and} \quad L_i(x_i) := \{ x_i' \in \mathbb{R}^\ell_+ \mid x_i \succsim_i x_i' \}.$$

A bundle x_i' is in the upper contour set of x_i if it is weakly preferred to x_i; it is in the lower contour set if it is weakly dispreferred to x_i. These sets are useful for defining some properties of preference relations. They are illustrated in Figure 7.1, assuming that $\succsim_i$ is increasing.

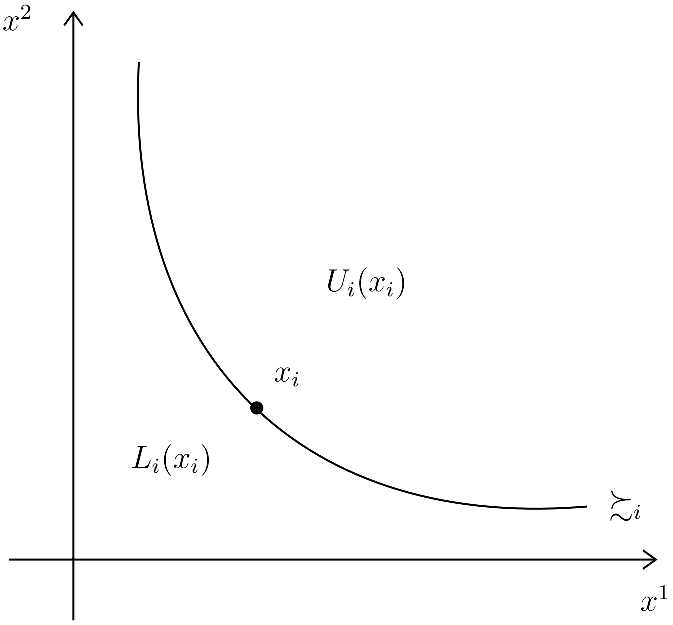

Figure 7.1: Upper and lower contour sets at bundle x_i.

Properties of preferences. As we did with preferences over lotteries, we now define some properties of preferences over consumption bundles. Some of the results we present later require these properties. Note that we are studying preferences over $\mathbb{R}_+^\ell$, which has a structure we can exploit. However, we no longer have lotteries, so we cannot use properties that rely on that structure, for example independence.[1]

Definition 7.3. *A preference relation $\succsim_i$ is **locally non-satiated** if, for every $x_i \in \mathbb{R}_+^\ell$ and every $\varepsilon > 0$, there exists $x_i' \in \mathbb{R}_+^\ell$ such that $\|x_i' - x_i\| < \varepsilon$ and $x_i' \succ_i x_i$.*

Local non-satiation says that in every neighbourhood of every bundle there is another bundle that is strictly preferred. It is a weak form of monotonicity. In fact, monotonicity implies Local non-satiation, but the converse is not true. It rules out the case in which all goods are *bads*, in the sense that individuals do not like them. Local non-satiation also rules out thick indifference curves. Consider the preferences represented in Figure 7.2. These preferences are not locally non-satiated at bundle x_i, since in any neighbourhood of x_i there is no strictly preferred bundle.

[1]However, in general equilibrium under uncertainty the lottery structure is important, see Mas-Colell et al. (1995, ch. 19).

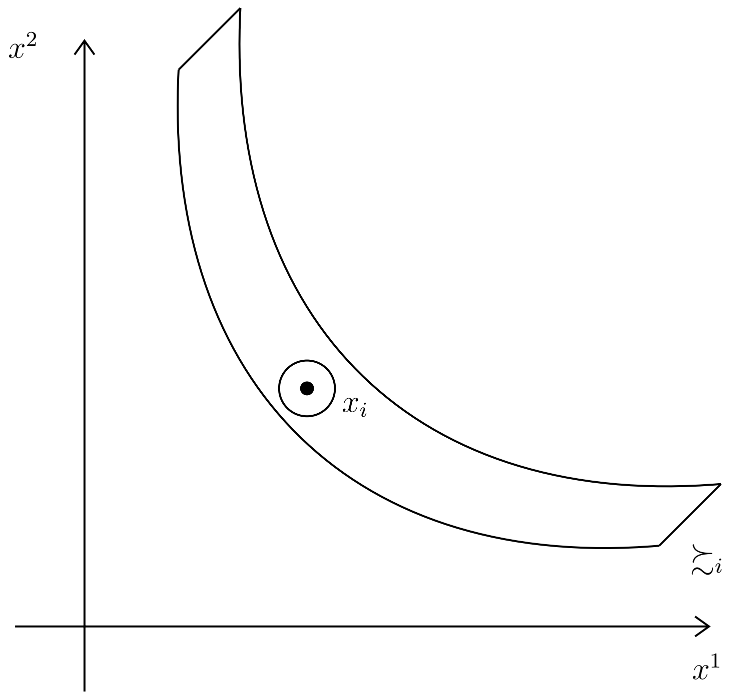

Figure 7.2: Preferences that are not locally non-satiated at bundle x_i.

A second important property is convexity.

Definition 7.4. *A preference relation $\succsim_i$ is **convex** if for all $x_i, x_i', x_i'' \in \mathbb{R}^\ell_+$, whenever $x_i' \succsim_i x_i$ and $x_i'' \succsim_i x_i$, then*

$$\alpha x_i' + (1-\alpha)x_i'' \succsim_i x_i \quad \textit{for all } \alpha \in [0,1].$$

Convexity is equivalent to upper contour sets being convex.[2] It captures a form of *diminishing marginal returns*. It can also be viewed as a preference for *diversification*.

Finally, we define continuity.

Definition 7.5. *A preference relation $\succsim_i$ on $\mathbb{R}^\ell_+$ is **continuous** if for every bundle $x_i \in \mathbb{R}^\ell_+$, both the upper and lower contour sets of x_i are closed.*

Continuity says that small changes in consumption bundles do not lead to jumps in preferences. If you are wondering about the relationship between this notion of continuity and the notion of Continuity over lotteries from Chapter 2, the answer is that the latter is *weaker* than the former. However, under the independence axiom for preferences over lotteries, they are equivalent.[3]

If a preference relation $\succsim_i$ is continuous, then for every endowment e_i and every strictly positive price vector $p \in \mathbb{R}^\ell_{++}$, the Walrasian demand $D_i(p, e_i)$ is non-empty. This is because a complete, transitive, and continuous preference relation admits a continuous utility representation (Mas-Colell et al., 1995, Proposition 3.C.1).[4] The budget

[2]You are asked to show this in Exercise 7.1.

[3]Actually, the notion of Archimedean continuity in Lecture 2 is equivalent to continuity in Definition 7.5 even under a specific weakening of independence (Karni, 2007).

[4]Interestingly, the result dates back to Cantor (1915); see the discussion in Gilboa (2009, ch. 6.1).

set $B(p, e_i)$ is compact when prices are strictly positive, so the continuous utility function attains a maximum on it. If preferences are also locally non-satiated, then the Walrasian demand $D_i(p, e_i)$ lies on the budget line $\{x_i \in \mathbb{R}^\ell_+ \mid p \cdot x_i = p \cdot e_i\}$.[5]

You might wonder whether there exists a reasonable-looking preference relation that is not continuous. The answer is yes. Consider the **lexicographic** preference relation on $\mathbb{R}^2_+$: for any two bundles $x_i = (x_i^1, x_i^2)$ and $x_i' = (x_i'^1, x_i'^2)$, define $x_i' \succsim_i x_i$ if either (i) $x_i'^1 > x_i^1$, or (ii) $x_i'^1 = x_i^1$ and $x_i'^2 \geq x_i^2$. This preference relation is complete and transitive, but it is not continuous. If you want, you can have fun showing this (or check Mas-Colell et al. (1995, p. 47)).

7.2 Allocations, rules and their properties

We have introduced the primitives of an exchange economy of interest. Given these primitives, we ask which allocations we can select in this economy. Such an allocation should have two types of properties: first, it should satisfy some normative criteria, for example efficiency or distributional fairness; second, it should be compatible with individuals' incentives, otherwise we could simply force an allocation while disregarding individual preferences.

The efficiency criterion we use is **Pareto optimality**.

Definition 7.6. *A feasible allocation x is* ***Pareto optimal*** *if there is no feasible allocation x' with $x_i' \succsim_i x_i$ for all i and $x_j' \succ_j x_j$ for some j.*

Pareto optimality says that a feasible allocation is Pareto optimal if there is no other feasible allocation that makes everyone weakly better off and at least one individual strictly better off.

The distributional criterion we consider is **no-envy**.

Definition 7.7. *An allocation x satisfies* ***no-envy*** *if for all individuals i and j,*

$$x_i \succsim_i x_j.$$

An allocation satisfies No-envy if no individual prefers the bundle of another individual to his own bundle.

For individual incentives, we consider **Walrasian equilibrium**.

Definition 7.8. *A feasible allocation x is a* ***Walrasian equilibrium*** *if there exist strictly positive prices $p \in \mathbb{R}^\ell_{++}$ such that, for all individuals i,*

$$x_i \in D_i(p, e_i).$$

[5]You are asked to show this in Exercise 7.2.

An allocation is a Walrasian equilibrium if there exist prices such that each individual would consume a bundle in his Walrasian demand given his endowment and those prices. Notice that we are not saying that these prices exist and that individuals actually face them. Instead, we are detailing a property of an allocation: can such an allocation be, in principle, the result of an individual optimisation problem in the style of choice theory? If yes, then the allocation is a Walrasian equilibrium.

We will consider a couple of variants of Walrasian equilibrium. A **Walrasian equilibrium with transfers** is a generalisation in which we allow transfers among individuals before they optimise.

Definition 7.9. *A feasible allocation x is a* ***Walrasian equilibrium with transfers*** *if there exist strictly positive prices $p \in \mathbb{R}^{\ell}_{++}$ and transfers $(T_i)_{i \in I}$ satisfying*

$$\sum_i T_i = 0 \quad \text{and} \quad e_i + T_i \in \mathbb{R}^{\ell}_{+} \text{ for all } i,$$

such that

$$x_i \in D_i(p, e_i + T_i) \text{ for all } i.$$

An allocation is a Walrasian equilibrium with transfers if there exist prices and transfers such that each individual would consume a bundle in his Walrasian demand given his adjusted endowment and those prices. Each Walrasian equilibrium is also a Walrasian equilibrium with transfers with no transfers, that is, $T_i = 0$ for all i.

The last equilibrium notion we consider is **Egalitarian Walrasian equilibrium**.

Definition 7.10. *A feasible allocation x is an* ***Egalitarian Walrasian equilibrium*** *if there exist strictly positive prices $p \in \mathbb{R}^{\ell}_{++}$ such that*

$$x_i \in D_i\left(p, \frac{\bar{e}}{n}\right) \text{ for all } i.$$

An allocation is a Egalitarian Walrasian equilibrium if there exist prices such that each individual would consume a bundle in his Walrasian demand given an equal share of the total endowment and those prices. Suppose we collect all the resources in the economy and redistribute them equally among individuals before they optimise. If the resulting allocation is a Walrasian equilibrium, it is a Egalitarian Walrasian equilibrium.

Let $F(E)$ be the set of feasible allocations. Which feasible allocations do we want to select? To answer this question, we introduce allocation rules. An **allocation rule** R maps an economy E to a subset of feasible allocations $R(E) \subseteq F(E)$. The standard axiomatic approach is to introduce assumptions on R and see what allocations it induces. For example, we might impose that R only selects Pareto optimal allocations. In what follows, we study allocation rules that select allocations with the efficiency, distributional, and incentive properties we introduced above.

Definition 7.11. *An allocation rule R^W is* ***Walrasian*** *if for all economies E it selects allocations that are Walrasian equilibria; that is, if for all E*

$$R^W(E) = \left\{x \in F(E) \;\middle|\; \exists p \in \mathbb{R}^\ell_{++} \ s.t.\ x_i \in D_i(p, e_i) \ for\ all\ i\right\}.$$

An allocation rule is Walrasian if it selects all Walrasian equilibria of the economy. Equivalently, we can define allocation rules that select all Walrasian equilibria with transfers and all Egalitarian Walrasian equilibria.

Definition 7.12. *An allocation rule R^{WT} is* ***Walrasian with transfers*** *if for all economies E it selects allocations that are Walrasian equilibria with transfers; that is, if for all E*

$$R^{WT}(E) = \left\{ x \in F(E) \;\middle|\; \begin{array}{ll} \exists p \in \mathbb{R}^\ell_{++},\ \exists (T_i)_i \ s.t. & e_i + T_i \in \mathbb{R}^\ell_+ \ for\ all\ i,\ \sum_i T_i = 0 \\ & x_i \in D_i(p, e_i + T_i) \ for\ all\ i \end{array} \right\}.$$

Definition 7.13. *An allocation rule R^{EW} is* ***Egalitarian Walrasian*** *if for all economies E it selects allocations that are Egalitarian Walrasian equilibria; that is, if for all E*

$$R^{EW}(E) = \left\{x \in F(E) \;\middle|\; \exists p \in \mathbb{R}^\ell_{++} \ s.t.\ x_i \in D_i\Big(p, \frac{\bar{e}}{n}\Big) \ for\ all\ i\right\}.$$

In the next lectures, we will study properties of these allocation rules and the relationships between them.

A question you might ask to better understand these definitions is: what exactly are allocations? It is easier to think first about a static problem, where allocations are just consumption bundles. However, in principle, an allocation might encode time and uncertainty. A consumption bundle might be a stream of consumption over time, or a contingent consumption plan over states of the world. We are simply not making this structure explicit. A deeper discussion of this point is in Debreu (1959, ch. 2).

Things to read. Properties of preferences are discussed in detail in Mas-Colell et al. (1995, ch. 3). Exchange economies are introduced in Hildenbrand & Kirman (1976, ch. 2). You can also read Mas-Colell et al. (1995, ch. 16) if you wish, but it moves straight to production economies. Varian (1992, ch. 17) instead studies exchange economies in detail. A treatment close to the approach in these notes is in Thomson (2011). A brief discussion of various Egalitarian Walrasian allocation rules is in Fleurbaey & Maniquet (2011, ch. 1). Debreu (1959) and Arrow & Hahn (1971) are classic references with historical value. You can find everything we discuss there.

7.3 Exercises

Exercise 7.1. Show that convexity in Definition 7.4 is equivalent to the following statement: for every bundle $x_i \in \mathbb{R}^\ell_+$, both the upper and lower contour sets of x_i are convex.

Solution to Exercise 7.1. We show the two directions of the equivalence.

($\Rightarrow$) Suppose that $\succsim_i$ is convex. Let $x_i \in \mathbb{R}^\ell_+$ and consider two bundles $x_i', x_i'' \in U_i(x_i)$. By definition of upper contour set, we have $x_i' \succsim_i x_i$ and $x_i'' \succsim_i x_i$. By convexity of $\succsim_i$, for any $\alpha \in [0, 1]$, we have

$$\alpha x_i' + (1 - \alpha)x_i'' \succsim_i x_i,$$

which implies that $\alpha x_i' + (1 - \alpha)x_i'' \in U_i(x_i)$. Thus, $U_i(x_i)$ is convex.

Now consider two bundles $y_i', y_i'' \in L_i(x_i)$. By definition of lower contour set, we have $x_i \succsim_i y_i'$ and $x_i \succsim_i y_i''$. By convexity of $\succsim_i$, for any $\alpha \in [0, 1]$, we have

$$x_i \succsim_i \alpha y_i' + (1 - \alpha)y_i'',$$

which implies that $\alpha y_i' + (1 - \alpha)y_i'' \in L_i(x_i)$. Thus, $L_i(x_i)$ is convex.

($\Leftarrow$) Suppose that for every bundle $x_i \in \mathbb{R}^\ell_+$, both the upper and lower contour sets of x_i are convex. Let $x_i, x_i', x_i'' \in \mathbb{R}^\ell_+$ such that $x_i' \succsim_i x_i$ and $x_i'' \succsim_i x_i$. This means that $x_i', x_i'' \in U_i(x_i)$. □

Exercise 7.2. Show that if a preference relation $\succsim_i$ is locally non-satiated, then for every endowment e_i and every strictly positive price vector $p \in \mathbb{R}^\ell_{++}$, the Walrasian demand $D_i(p, e_i)$ lies on the budget line $\{x_i \in \mathbb{R}^\ell_+ \mid p \cdot x_i = p \cdot e_i\}$.

Solution to Exercise 7.2. Suppose that $\succsim_i$ is locally non-satiated. Let $e_i \in \mathbb{R}^\ell_+$ be an endowment and $p \in \mathbb{R}^\ell_{++}$ be a strictly positive price vector. Consider a bundle $x_i \in D_i(p, e_i)$. By definition of Walrasian demand, we have $x_i \in B(p, e_i)$, which implies that $p \cdot x_i \leq p \cdot e_i$.

Suppose, for the sake of contradiction, that $p \cdot x_i < p \cdot e_i$. Since p is strictly positive, there exists $\varepsilon > 0$ such that the ball $B_\varepsilon(x_i) = \{x_i' \in \mathbb{R}^\ell_+ \mid \|x_i' - x_i\| < \varepsilon\}$ contains a bundle x_i' with $p \cdot x_i' < p \cdot e_i$ and $x_i' \succ_i x_i$, by local non-satiation of $\succsim_i$. This contradicts the fact that $x_i \in D_i(p, e_i)$, since $x_i' \in B(p, e_i)$ and $x_i' \succ_i x_i$.

Therefore, it must be that $p \cdot x_i = p \cdot e_i$. Thus, the Walrasian demand $D_i(p, e_i)$ lies on the budget line $\{x_i \in \mathbb{R}^\ell_+ \mid p \cdot x_i = p \cdot e_i\}$. □

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Lecture 8

First theorem of welfare economics

In this lecture we discuss the first theorem of welfare economics, which states that every Walrasian equilibrium with transfers is Pareto optimal. We now start seeing proofs that are a bit more sophisticated.[1] In particular, we will make use of proofs *by contradiction*. The logic is as follows. We want to prove a statement S. We assume that S is false and show that this assumption leads to a contradiction: for instance, we derive a conclusion of the form C and **not** C, or we obtain a conclusion that contradicts one of the premises needed for S. Either way, the assumption that S is false cannot be sustained, so S must be true.

We begin with a preliminary lemma.

Lemma 8.1. *Assume that preferences $\succsim_i$ are locally non-satiated. Let $x_i \in D_i(p, e_i)$. If $x_i' \succsim_i x_i$, then $p \cdot x_i' \geq p \cdot x_i$.*

Proof. Suppose, to the contrary, that $p \cdot x_i' < p \cdot x_i$. Let

$$\delta = \frac{p \cdot x_i - p \cdot x_i'}{2} > 0.$$

By Local non-satiation, for every $\varepsilon > 0$ there exists $x_i'' \in \mathbb{R}^\ell_+$ such that $\|x_i'' - x_i'\| < \varepsilon$ and $x_i'' \succ_i x_i'$.

Choose ε small enough so that $\|x_i'' - x_i'\| < \varepsilon$ implies $|p \cdot x_i'' - p \cdot x_i'| < \delta$. Then

$$p \cdot x_i'' \leq p \cdot x_i' + \delta < p \cdot x_i.$$

Since $x_i \in D_i(p, e_i)$, we have $p \cdot x_i \leq p \cdot e_i$, hence $p \cdot x_i'' < p \cdot e_i$ and therefore $x_i'' \in B(p, e_i)$. Moreover, $x_i'' \succ_i x_i'$ and $x_i' \succsim_i x_i$ imply $x_i'' \succ_i x_i$ by transitivity. This contradicts the fact that $x_i \in D_i(p, e_i)$. □

And now the main result.

Theorem 8.1. ***(First theorem of welfare economics)*** *If all preferences in economy E are locally non-satiated, then every allocation selected by R^{WT} is Pareto optimal. That is,*

$$x \in R^{WT}(E) \implies x \text{ is Pareto optimal.}$$

Proof. Let E be an economy and suppose that $x \in R^{WT}(E)$. Then there exist strictly positive prices p and transfers $(T_i)_i$ such that, for every individual i,

[1] An immensely valuable resource to learn fundamental proof strategies is Cummings (2021). If you want to have fun and understand Italian, you can read Lolli (2020).

$$x_i \in D_i(p, e_i + T_i),$$

that is, x_i is a most preferred bundle in the budget set $B(p, e_i + T_i)$.

Suppose, for a contradiction, that x is not Pareto optimal. Then there exists a feasible allocation x' such that $x'_i \succsim_i x_i$ for all i, and $x'_j \succ_j x_j$ for some individual j. By Lemma 8.1, this implies

$$p \cdot x'_i \geq p \cdot x_i \quad \text{for all } i.$$

Moreover, for the individual j with $x'_j \succ_j x_j$ we in fact have $p \cdot x'_j > p \cdot x_j$: if $p \cdot x'_j \leq p \cdot x_j$, then $x'_j \in B(p, e_j + T_j)$, since $x_j \in B(p, e_j + T_j)$, contradicting $x_j \in D_j(p, e_j + T_j)$. Therefore,

$$p \cdot x'_j > p \cdot x_j \quad \text{for some } j.$$

Summing over all individuals,

$$\sum_i p \cdot x'_i > \sum_i p \cdot x_i. \tag{*}$$

On the other hand, local non-satiation implies that each demanded bundle exhausts the budget, so $p \cdot x_i = p \cdot (e_i + T_i)$ for all i. Summing and using $\sum_i T_i = 0$,

$$\sum_i p \cdot x_i \;=\; \sum_i p \cdot (e_i + T_i) \;=\; \sum_i p \cdot e_i.$$

Since x' is feasible, $\sum_i x'_i \leq \sum_i e_i$, and multiplying by the strictly positive price vector p gives

$$\sum_i p \cdot x'_i \;\leq\; \sum_i p \cdot e_i \;=\; \sum_i p \cdot x_i,$$

contradicting $(*)$. Hence no such x' exists, and x is Pareto optimal. $\square$

Before turning to the interpretation of Theorem 8.1, let us discuss why Local non-satiation is necessary for the theorem to hold. Look at Figure 8.1. Individual 1 has "thick" indifference curves, so he violates local non-satiation. The allocation x is not Pareto optimal, since there is another feasible allocation x' that makes individual 2 strictly better off without making individual 1 worse off. However, x can still be supported as a Walrasian equilibrium with transfers at prices p. Thus, without local non-satiation, Theorem 8.1 fails.

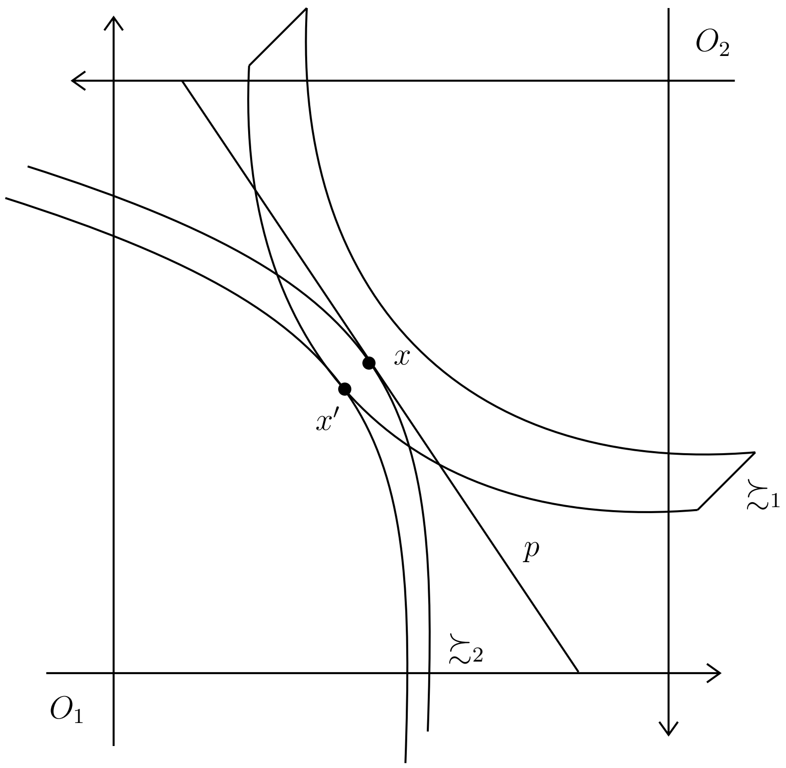

Figure 8.1: Local non-satiation is necessary for the First Welfare Theorem.

Why can x be supported as a Walrasian equilibrium with transfers in Figure 8.1?

Let us now interpret Theorem 8.1. A classical economic question is how to allocate resources given information about preferences.[2] We might want such an allocation of resources to satisfy some attractive properties from a *normative* perspective. One such property is Pareto optimality. An allocation rule that takes preferences and endowments as inputs and returns a Pareto optimal allocation as output might therefore be desirable. However, once one has an allocation rule, one might wonder whether it can be implemented in a decentralised way. A rule that maps preferences and endowments into an allocation does not, by itself, tell us *how* to reach that allocation.

Theorem 8.1 tells us that Walrasian equilibrium with transfers implements an allocation rule that always delivers Pareto optimal allocations. Decentralised individual optimisation at some prices can therefore lead to desirable allocations from this point of view. There is a bit more, however. A Walrasian equilibrium with transfers is not just a mechanism to implement Pareto optimal allocations. It also induces a price for each good, which individuals can use to trade in order to reach those allocations. Prices may be interpreted as "values" for goods, where these values depend on the preferences and

[2]In its most general form, a question of this kind is asked in Arrow (2012), the spark that gave rise to modern social choice theory.

endowments of all individuals in the economy. In fact, Debreu (1959) is titled *Theory of Value.*

Many more or less sophisticated critiques of economics stem from the idea that it is inappropriate to view the value of goods as determined by prices. There is a famous quotation, often misattributed to Oscar Wilde,[3] that says: "An economist is someone who knows the price of everything and the value of nothing". Of course, there are legitimate reasons to question whether values should be entirely derived from individual preferences. But, in the setting we consider here, there is nothing special about prices that is not ultimately related to preferences or endowments.[4]

Unfortunately, an allocation rule that delivers Pareto optimal allocations can sometimes be undesirable from other points of view. For instance, it might deliver very unequal allocations. We might therefore want to complement Pareto optimality with a distributional requirement. One candidate is envy-freeness, which is closely related to the idea of equality of opportunity. One way to make this relationship precise is captured in the following proposition.

Proposition 8.1. *Every allocation selected by R^{EW} satisfies No-envy. That is,*

$$x \in R^{EW}(E) \quad \Longrightarrow \quad x \text{ satisfies No-envy.}$$

Proof. Let E be an economy and suppose that $x \in R^{EW}(E)$. By definition of Egalitarian Walrasian equilibrium, there exists a price vector $p \in \mathbb{R}^{\ell}_{++}$ such that, for each individual i,

$$x_i \in D_i\Big(p, \frac{\bar{e}}{n}\Big).$$

Because all individuals face the same endowment $\frac{\bar{e}}{n}$ and the same price vector p, they all face the same budget set

$$B := B\Big(p, \frac{\bar{e}}{n}\Big) = \left\{ x'_i \in \mathbb{R}^{\ell}_{+} \mid p \cdot x'_i \leq p \cdot \tfrac{\bar{e}}{n} \right\}.$$

Fix any individual i. Since $x_i \in D_i\big(p, \frac{\bar{e}}{n}\big)$, x_i is a most preferred bundle for i in B. Therefore,

$$\text{for all } x'_i \in B, \qquad x_i \succsim_i x'_i. \tag{8.1}$$

Now consider any other individual $j \neq i$. Since $x_j \in D_j\big(p, \frac{\bar{e}}{n}\big)$, we have $x_j \in B$. Applying (8.1) to the specific bundle x_j yields

[3] Apparently the original quotation is "[A cynic is] a man who knows the price of everything, and the value of nothing" from Wilde (1995, p. 55).

[4] However, sometimes the mere existence of prices, from a physical point of view, induces disgust towards the commodification of goods that "should not be priced". As Sophocles (1939, p. 201) puts it: "There's nothing in the world so demoralizing as money". Fleurbaey et al. (2025) studies a class of problems, of which commodification is one, in a general equilibrium setting, so you are ready to read it!

$$x_i \succsim_i x_j.$$

Since i and $j \neq i$ were arbitrary, it follows that for all $i \neq j$,

$$x_i \succsim_i x_j,$$

which is exactly No-envy. Hence the allocation x satisfies No-envy. $\square$

Since any Egalitarian Walrasian equilibrium is also a Walrasian equilibrium with transfers, take transfers $T_i := \frac{\bar{e}}{n} - e_i$, Theorem 8.1 and Proposition 8.1 together imply that the Egalitarian Walrasian allocation rule delivers allocations that are both Pareto optimal and satisfy No-envy. Therefore, in this simple setting, requirements of *efficiency*, *fairness*, and *incentives* are compatible![5]

However, the actual endowments e_i need not coincide with the egalitarian endowment $\frac{\bar{e}}{n}$. We might therefore be interested in understanding when such an allocation rule can be implemented starting from an arbitrary endowment profile. The second fundamental theorem of welfare economics, which we discuss in the next lecture, provides conditions under which this is possible.

Things to read. The proof of Theorem 8.1 in these notes follows Mas-Colell et al. (1995, pp. 545–550). A version withing exchange economies is in Varian (1992, ch. 17). A proof of a closely related version of Proposition 8.1 appears in Fleurbaey (2008, p. 46), which also discusses further properties of allocations satisfying No-envy.

8.1 Exercises

Exercise 8.1. Explain why Proposition 8.1 links envy-freeness to equality of opportunity. There are at least a couple of things to say here. For instance, do the allocations selected by the Egalitarian Walrasian allocation rule depend on individuals' endowments?

Solution to Exercise 8.1. An allocation x selected by the Egalitarian Walrasian allocation rule is supported as a Egalitarian Walrasian equilibrium at prices p with egalitarian endowments $\frac{\bar{e}}{n}$ for all individuals. Therefore, individuals face the same budget set $B(p, \frac{\bar{e}}{n})$. Since each individual chooses his most preferred bundle in that budget set, no individual prefers another individual's bundle to his own. Hence, no individual envies another individual.

The allocations selected by the Egalitarian Walrasian allocation rule do not depend on individuals' actual endowments e_i. They depend only on the total endowment $\bar{e}$ and the preferences of individuals. This reflects the idea of equality of opportunity: individuals

[5] Thomson (2011, p. 405) discusses that the Egalitarian Walrasian allocation rule is also easy to implement under incomplete information about preferences.

are given equal resources, the same budget set, to choose from, regardless of their initial endowments. $\square$

Exercise 8.2. In this exercise you should use calculus to compute Walrasian demands and equilibria. Feel free to use this tool. Consider an exchange economy with two individuals $i \in \{1,2\}$ and two goods $\ell \in \{1,2\}$. An allocation is $x = (x_1, x_2)$ with $x_i = (x_i^1, x_i^2) \in \mathbb{R}^2_+$, and endowments are $e_i = (e_i^1, e_i^2) \in \mathbb{R}^2_+$. Prices are $p = (p^1, p^2) \in \mathbb{R}^2_{++}$ and the budget set is $B(p, e_i) = \{x_i \in \mathbb{R}^2_+ \mid p \cdot x_i \le p \cdot e_i\}$.

Assume endowments are

$$e_1 = (1,0), \qquad e_2 = (0,1),$$

and preferences are represented by Cobb–Douglas utilities with different exponents:

$$u_1(x_1) = (x_1^1)^\alpha \,(x_1^2)^{1-\alpha}, \qquad u_2(x_2) = (x_2^1)^\beta \,(x_2^2)^{1-\beta},$$

where $\alpha, \beta \in (0,1)$ and $\alpha \neq \beta$.

1. Fix prices $p \in \mathbb{R}^2_{++}$. Compute each individual's Walrasian demand $D_i(p, e_i)$.
2. Find a Walrasian equilibrium: determine a price vector $p \in \mathbb{R}^2_{++}$ and a feasible allocation $x = (x_1, x_2)$ such that $x_i \in D_i(p, e_i)$ for $i = 1, 2$.
3. Is the Walrasian equilibrium allocation envy-free? If not, give a condition on (α, β) under which it becomes envy-free.

Solution to Exercise 8.2. 1. Fix prices $p \in \mathbb{R}^2_{++}$. Individual i solves

$$\max_{x_i \in \mathbb{R}^2_+} u_i(x_i) \qquad \text{s.t.} \qquad p \cdot x_i \le p \cdot e_i.$$

For Cobb–Douglas preferences with $\alpha, \beta \in (0,1)$, the optimum is interior, as both goods are desirable, and the budget constraint binds. Write $w_i := p \cdot e_i$ for wealth.

Agent 1. Consider the Lagrangian

$$\mathcal{L}_1(x_1^1, x_1^2, \lambda_1) = (x_1^1)^\alpha (x_1^2)^{1-\alpha} + \lambda_1\big(w_1 - p^1 x_1^1 - p^2 x_1^2\big).$$

The first-order conditions are

$$\frac{\partial \mathcal{L}_1}{\partial x_1^1} = \alpha (x_1^1)^{\alpha-1} (x_1^2)^{1-\alpha} - \lambda_1 p^1 = 0,$$

$$\frac{\partial \mathcal{L}_1}{\partial x_1^2} = (1-\alpha)(x_1^1)^\alpha (x_1^2)^{-\alpha} - \lambda_1 p^2 = 0,$$

together with complementary slackness, which here reduces to the binding budget constraint

$$p^1 x_1^1 + p^2 x_1^2 = w_1.$$

Divide the first two FOCs to eliminate λ_1:

$$\frac{\alpha(x_1^1)^{\alpha-1}(x_1^2)^{1-\alpha}}{(1-\alpha)(x_1^1)^{\alpha}(x_1^2)^{-\alpha}} = \frac{p^1}{p^2} \quad\iff\quad \frac{\alpha}{1-\alpha}\cdot\frac{x_1^2}{x_1^1} = \frac{p^1}{p^2}.$$

Rearranging gives a convenient expression of the optimal *ratio*:

$$x_1^2 = \frac{1-\alpha}{\alpha}\cdot\frac{p^1}{p^2}\,x_1^1.$$

Substitute this into the budget constraint:

$$p^1x_1^1 + p^2\left(\frac{1-\alpha}{\alpha}\cdot\frac{p^1}{p^2}\,x_1^1\right) = w_1 \quad\iff\quad p^1x_1^1\left(1+\frac{1-\alpha}{\alpha}\right) = w_1.$$

Since $1+\frac{1-\alpha}{\alpha} = \frac{1}{\alpha}$, we obtain

$$p^1x_1^1\cdot\frac{1}{\alpha} = w_1 \quad\iff\quad x_1^1 = \frac{\alpha w_1}{p^1}.$$

Finally, plug back into the ratio, or directly into the budget constraint, to get

$$x_1^2 = \frac{(1-\alpha)w_1}{p^2}.$$

Hence the Walrasian demand of agent 1 is

$$D_1(p,e_1) = \Big(\frac{\alpha w_1}{p^1},\ \frac{(1-\alpha)w_1}{p^2}\Big).$$

Agent 2. The calculation is identical. The Lagrangian is

$$\mathcal{L}_2(x_2^1,x_2^2,\lambda_2) = (x_2^1)^{\beta}(x_2^2)^{1-\beta} + \lambda_2\big(w_2 - p^1x_2^1 - p^2x_2^2\big),$$

and the FOCs yield

$$D_2(p,e_2) = \Big(\frac{\beta w_2}{p^1},\ \frac{(1-\beta)w_2}{p^2}\Big).$$

In our economy $e_1=(1,0)$ and $e_2=(0,1)$, hence $w_1 = p\cdot e_1 = p^1$ and $w_2 = p\cdot e_2 = p^2$. Substituting,

$$D_1(p,e_1) = \Big(\alpha,\ (1-\alpha)\frac{p^1}{p^2}\Big), \qquad D_2(p,e_2) = \Big(\beta\frac{p^2}{p^1},\ 1-\beta\Big).$$

2. A Walrasian equilibrium is a price vector $p\in\mathbb{R}^2_{++}$ and a feasible allocation $x=(x_1,x_2)$ such that $x_i\in D_i(p,e_i)$ and markets clear.

   Market clearing for good 1 requires

$$x_1^1 + x_2^1 = 1.$$

Using the demands from part (1),

$$\alpha + \beta\frac{p^2}{p^1} = 1 \quad \Longleftrightarrow \quad \frac{p^2}{p^1} = \frac{1-\alpha}{\beta}.$$

Thus equilibrium prices are determined only up to a positive normalization; what matters is the relative price. A convenient choice is

$$p = (p^1, p^2) = (\beta,\, 1-\alpha),$$

which satisfies $\frac{p^2}{p^1} = \frac{1-\alpha}{\beta}$.

Given this relative price, the equilibrium allocation is obtained by plugging into demands:

$$x_1 = D_1(p, e_1) = \Big(\alpha,\, (1-\alpha)\frac{p^1}{p^2}\Big) = \Big(\alpha,\, (1-\alpha)\frac{\beta}{1-\alpha}\Big) = (\alpha, \beta),$$

and

$$x_2 = D_2(p, e_2) = \Big(\beta\frac{p^2}{p^1},\, 1-\beta\Big) = \Big(\beta\frac{1-\alpha}{\beta},\, 1-\beta\Big) = (1-\alpha,\, 1-\beta).$$

Feasibility is immediate:

$$x_1 + x_2 = (\alpha + 1 - \alpha,\;\; \beta + 1 - \beta) = (1, 1) = e_1 + e_2.$$

3. Envy-freeness means that no one strictly prefers the other person's bundle:

$$u_1(x_1) \geq u_1(x_2) \quad \text{and} \quad u_2(x_2) \geq u_2(x_1).$$

Here $x_1 = (\alpha, \beta)$ and $x_2 = (1-\alpha, 1-\beta)$, so these inequalities become

$$\begin{aligned} \alpha^\alpha \beta^{1-\alpha} \;&\geq\; (1-\alpha)^\alpha (1-\beta)^{1-\alpha}, \\ (1-\alpha)^\beta (1-\beta)^{1-\beta} \;&\geq\; \alpha^\beta \beta^{1-\beta}. \end{aligned}$$

Equivalently, taking logs, one can write them as two linear inequalities in $\ln\frac{\alpha}{1-\alpha}$ and $\ln\frac{\beta}{1-\beta}$.

In general the Walrasian equilibrium allocation need not be envy-free. A simple and sharp-looking *sufficient* condition that guarantees envy-freeness is

$$\alpha + \beta = 1.$$

Indeed, if $\beta = 1 - \alpha$, then

$$x_1 = (\alpha, 1-\alpha), \qquad x_2 = (1-\alpha, \alpha).$$

Agent 1 puts weight α on good 1 and $1-\alpha$ on good 2, so he weakly prefers the bundle that loads more on good 1 precisely when $\alpha \geq \frac{1}{2}$; but in that case x_1 is exactly the bundle with more of good 1. Symmetrically, agent 2 has exponent $\beta = 1 - \alpha$, so he weakly prefers the bundle that loads more on good 1 precisely when $1 - \alpha \geq \frac{1}{2}$, and in that case x_2 is exactly the bundle with more of good 1. Hence neither envies the other.

□

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Lecture 9

Second theorem of welfare economics

We now turn to the second theorem of welfare economics, which establishes that, under certain conditions, any Pareto optimal allocation can be implemented as a Walrasian equilibrium with transfers. The proof relies on convexity of preferences and uses a geometric argument based on supporting hyperplanes. Supporting and separating hyperplane arguments are common in economic theory.[1] In particular, we will invoke the specific result stated below.[2] Recall that the interior of a set $X \subset \mathbb{R}^\ell$, denoted $\operatorname{int} X$, is the set of all points $x \in X$ for which there exists an open ball around x that is fully contained in X.

Theorem 9.1. ***(Supporting hyperplane)*** *Let $X \subset \mathbb{R}^\ell$ be a convex set and let $x \notin \operatorname{int} X$. Then there exists $p \in \mathbb{R}^\ell$ with $p \neq 0$ such that*

$$p \cdot x \;\geq\; p \cdot y \qquad \text{for all } y \in X.$$

In words, there is a hyperplane with normal vector p passing through x such that X lies entirely in the weak half-space on one side of it. Figure 9.1 illustrates the theorem in two dimensions.

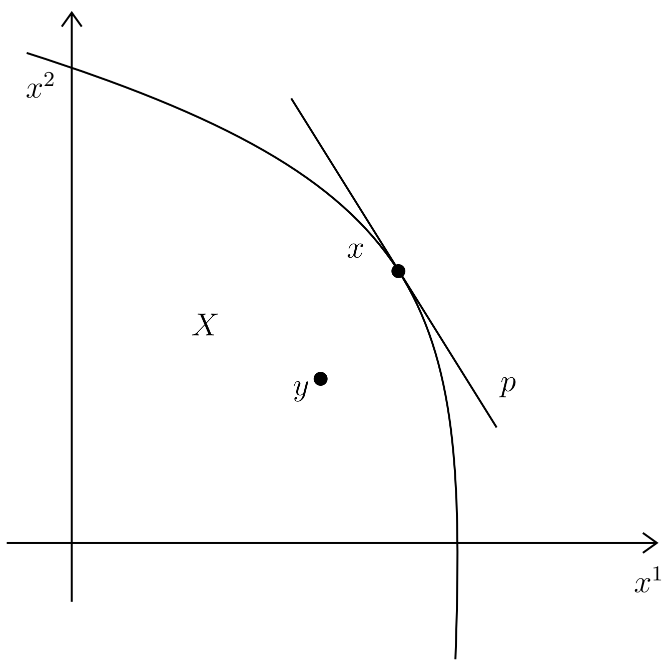

Figure 9.1: A supporting hyperplane through x for the convex set X.

[1] As an example, a proof of the expected utility representation in Lecture 2 using a separation argument is in Gilboa (2009, ch. 8.3.3).

[2] If you are interested, a nice source is Rockafellar (1970, ch. 11).

The last ingredient we need is a strengthening of .

Definition 9.1. *A preference relation $\succsim_i$ on $\mathbb{R}^\ell_+$ is* ***monotonic*** *if for every $x_i, x_i' \in \mathbb{R}^\ell_+$ with $x_i' \geq x_i$ and $x_i' \neq x_i$, we have $x_i' \succ_i x_i$.*

A monotonic preference relation is one where (holding everything else fixed) more of any good is always strictly preferred to less. Monotonicity implies local non-satiation, but the converse is not true.

We can now state and prove the second fundamental theorem of welfare economics.

Theorem 9.2. ***(Second welfare theorem)*** *If all preferences in the economy E are monotonic, convex, and continuous, and each individual has a strictly positive endowment $e_i \in \mathbb{R}^\ell_{++}$, then every interior Pareto optimal allocation $x \in \mathbb{R}^{\ell n}_{++}$ can be supported as a Walrasian equilibrium with transfers. That is,*

$$x \text{ is Pareto optimal and } x \in \mathbb{R}^{\ell n}_{++} \quad \Longrightarrow \quad x \in R^{WT}(E).$$

Proof. Say we want to implement the interior Pareto optimal allocation x. Define transfers T_i by

$$T_i = x_i - e_i \qquad \text{for each } i.$$

Feasibility of x implies $\sum_i x_i = \sum_i e_i$, hence $\sum_i T_i = 0$, so $(T_i)_i$ is a feasible vector of lump–sum transfers. We have to find strictly positive prices p such that for each individual, endowed with $e_i + T_i = x_i$, the bundle x_i is in the Walrasian demand of i:

$$x_i \in D_i(p, e_i + T_i) \qquad \text{for each } i.$$

Step 1: "Strictly better–than" sets. For each individual i, let

$$\overline{U}^i(x_i) := \{x_i' : x_i' \succ_i x_i\}$$

denote the **strict** upper contour set at x_i. By continuity and convexity of $\succsim_i$ this set is convex, and $x_i \notin \overline{U}^i(x_i)$. Define the set of aggregate improvements

$$\overline{U}(x) = \sum_i \overline{U}^i(x_i) := \left\{x' \;\middle|\; x' = \sum_i x_i' \text{ with } x_i' \in \overline{U}^i(x_i) \text{ for each } i\right\}.$$

So $\overline{U}(x)$ is the set of all aggregate bundles that can be obtained by letting each individual choose a bundle strictly preferred to her allocation in x. Since it is the sum of convex sets, $\overline{U}(x)$ is convex. Pareto optimality of x says that there is no **feasible** allocation $x' = (x_i')_i$ with $x_i' \succ_i x_i$ for all i. Equivalently,

$$\sum_i x_i \notin \overline{U}(x).$$

Step 2: A supporting price hyperplane. We have a convex set $\overline{U}(x)$ and a point $\sum_i x_i$ outside it. By Theorem 9.1, there exists a nonzero vector p such that[3]

$$p \cdot x' \;\geq\; p \cdot \Big(\sum_i x_i\Big) \qquad \text{for all } x' \in \overline{U}(x).$$

In particular, for any profile $(x_i')_i$ with $x_i' \in \overline{U}^i(x_i)$ for all i,

$$\sum_i p \cdot x_i' = p \cdot \Big(\sum_i x_i'\Big) \;\geq\; p \cdot \Big(\sum_i x_i\Big) = \sum_i p \cdot x_i. \tag{9.1}$$

We need to show that prices are strictly positive and that each x_i is optimal in individual i's budget set at prices p and income $p \cdot (e_i + T_i) = p \cdot x_i$.

Step 3: Nonnegativity of supporting prices. Let $\omega := \sum_i x_i$ denote the aggregate bundle at the Pareto optimal allocation. For each good $k \in \{1, \ldots, \ell\}$, let $\mathbf{e}^k \in \mathbb{R}^\ell$ be the k-th unit vector.

Consider the allocation $\widehat{x}$ defined by

$$\widehat{x}_i := x_i + \frac{1}{n}\mathbf{e}^k \qquad \text{for each } i.$$

By monotonicity, $\widehat{x}_i \succ_i x_i$ for every i, hence $\sum_i \widehat{x}_i = \omega + \mathbf{e}^k \in \overline{U}(x)$. Applying Equation (9.1) to the profile $(\widehat{x}_i)_i$ gives

$$p \cdot (\omega + \mathbf{e}^k) \;\geq\; p \cdot \omega,$$

hence $p \cdot \mathbf{e}^k \geq 0$, i.e. $p^k \geq 0$. Since k was arbitrary, $p \in \mathbb{R}^\ell_+$.

Step 4: Any strictly preferred bundle must cost strictly more. Fix an individual j. We claim that if $x_j^{'} \succ_j x_j$, then

$$p \cdot x_j^{'} \;>\; p \cdot x_j.$$

Step 4(a): weak inequality. Suppose $x_j^{'} \succ_j x_j$. By continuity of $\succsim_j$, there exists $\theta \in (0,1)$ close enough to 0 such that $(1-\theta)x_j^{'} \succ_j x_j$. Define a profile x' by

$$x_j' := (1-\theta)x_j^{'}, \qquad x_i' := x_i + \frac{\theta}{n-1}x_j^{'} \quad \text{for all } i \neq j.$$

By monotonicity, for each $i \neq j$ we have $x_i' \succ_i x_i$, and by construction $x_j' \succ_j x_j$. Hence $\sum_i x_i' \in \overline{U}(x)$, and therefore (9.1) implies

$$p \cdot \Big(\sum_i x_i'\Big) \;\geq\; p \cdot \Big(\sum_i x_i\Big).$$

But $\sum_i x_i' = x_j^{'} + \sum_{i \neq j} x_i$, so cancelling $\sum_{i \neq j} p \cdot x_i$ yields

[3] Reversing the inequality does not affect Theorem 9.1.

$$p \cdot x_j' \;\geq\; p \cdot x_j.$$

Step 4(b): strict inequality. Assume for contradiction that $p \cdot x_j' = p \cdot x_j$. Since $x_j' \succ_j x_j$ and preferences are continuous, we can pick $\alpha \in (0,1)$ close enough to 1 so that $\alpha x_j' \succ_j x_j$. Applying Step 4(a) to $\alpha x_j'$ gives

$$p \cdot (\alpha x_j') \;\geq\; p \cdot x_j.$$

Using $p \cdot x_j' = p \cdot x_j$, this becomes $\alpha\, p \cdot x_j \geq p \cdot x_j$, hence $p \cdot x_j \leq 0$. But $x_j \in \mathbb{R}^\ell_{++}$ and $p \in \mathbb{R}^\ell_{+}$ with $p \neq 0$ imply $p \cdot x_j > 0$, a contradiction. Therefore $p \cdot x_j' > p \cdot x_j$.

Step 5: Strict positivity and individual optimality. *Strict positivity.* Fix any good k. For any $\varepsilon > 0$, monotonicity implies $x_j + \varepsilon \mathbf{e}^k \succ_j x_j$. By Step 4,

$$p \cdot (x_j + \varepsilon \mathbf{e}^k) \;>\; p \cdot x_j \quad \Rightarrow \quad \varepsilon p^k > 0 \quad \Rightarrow \quad p^k > 0.$$

Since k was arbitrary, $p \in \mathbb{R}^\ell_{++}$.

Optimality. Fix i. Suppose there exists $x_i' \in \mathbb{R}^\ell_{+}$ with $p \cdot x_i' \leq p \cdot x_i$ and $x_i' \succ_i x_i$. This contradicts Step 4, which says $x_i' \succ_i x_i \Rightarrow p \cdot x_i' > p \cdot x_i$. Hence no strictly preferred bundle is affordable at income $p \cdot x_i$, so x_i is optimal in the budget set:

$$x_i \in D_i(p, e_i + T_i).$$

Conclusion. We have found strictly positive prices $p \in \mathbb{R}^\ell_{++}$ and transfers $(T_i)_i$ such that

$$x_i \in D_i(p, e_i + T_i) \qquad \text{for all } i.$$

Hence x is a Walrasian equilibrium with transfers: $x \in R^{WT}(E)$. Since x was an arbitrary interior Pareto optimal allocation, the theorem follows. □

Let us now take stock of what we have achieved so far. The first fundamental theorem of welfare economics, Theorem 8.1, says that every Walrasian equilibrium allocation is Pareto optimal. The second fundamental theorem of welfare economics, Theorem 9.2, says that, under stronger assumptions, every interior Pareto optimal allocation can be implemented as a Walrasian equilibrium with transfers. Together, these two theorems establish a strong link between competitive equilibria and efficiency.

Things to read. This lecture is based on Varian (1992, ch. 17). For a treatment that includes production, see Mas-Colell et al. (1995, pp. 545–550).

9.1 Exercises

Exercise 9.1. If one **assumes** the existence of a Walrasian equilibrium with transfers, an indirect proof of the second welfare theorem can be given. Prove the following statement. Suppose that all preferences in the economy are locally non-satiated and that x^* is a Pareto optimal allocation. Suppose further that a Walrasian equilibrium exists when endowments are $e_i = x_i^*$ for all i. Then x^* can be supported as a Walrasian equilibrium allocation with transfers. (Hint: if you are stuck, see Varian (1992, p. 329).)

References

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Lecture 10

Existence of competitive equilibria

We have defined Walrasian (competitive) equilibrium and discussed its properties. In this lecture, we will prove that a competitive equilibrium exists under certain assumptions on preferences. We assume preferences are continuous, strictly convex, and strongly monotone. These are strengthening of assumptions we have made before.

Definition 10.1. *A preference relation is* ***strongly monotone*** *if for all $x, y \in \mathbb{R}^L_+$ such that $x \geq y$ and $x \neq y$, we have $x \succ y$.*

Definition 10.2. *A preference relation is* ***strictly convex*** *if for all $x, y, z \in \mathbb{R}^L_+$ such that $y \succ x$ and $z \succ x$, we have for all $\alpha \in (0, 1)$, $\alpha y + (1 - \alpha)z \succ x$. Equivalently, all upper contour sets are strictly convex sets.*

If a preference relation is continuous, strictly convex, and strongly monotone, then the Walrasian demand is single-valued and can therefore be viewed as a function. We can then define the excess demand function as follows.

Definition 10.3. *The* ***excess demand function*** *of and individual i with a single-valued Walrasian demand function $D_i(p, e_i)$ is given by*

$$z_i(p, e_i) = D_i(p, e_i) - e_i.$$

From the individual excess demand functions, we can construct the aggregate excess demand function.

$$z(p) = \sum_i z_i(p).$$

The excess demand function maps prices to allocations. Under our assumptions on preferences, Walrasian equilibrium can be characterised through the excess demand function as follows.

Proposition 10.1. *If individual preferences $\succsim_i$ are continuous, strictly convex, and strongly monotone for each i, then an allocation x is a Walrasian equilibrium if and only if there exists a price vector p such that $z(p) = 0$.*

The excess demand function has several important properties that we will use to prove the existence of a competitive equilibrium.

Proposition 10.2. *If individual preferences $\succsim_i$ are continuous, strictly convex, and strongly monotone for each i, then the aggregate excess demand function $z(p)$ satisfies the following properties:*

1. *$z(p)$ is **homogeneous of degree zero**:$z(\alpha p) = z(p)$ for all $\alpha > 0$;*

2. *$z(p)$ satisfies Walras' law: $p \cdot z(p) = 0$ for all strictly positive prices p;*

3. *$z(p)$ is continuous;*

4. *there is an $s > 0$ such that for all goods ℓ and prices p, $z_\ell(p) > -s$;*

5. *if p^n is a sequence of prices converging to p with $p_\ell = 0$ for some good ℓ, then $\max\{z^1(p^n), \ldots, z^L(p^n)\} \to +\infty$.*

The existence result we study here relies on Kakutani's fixed–point theorem.

Theorem 10.1. ***(Kakutani's fixed point)*** *Let $A \subseteq \mathbb{R}^L$ be a non-empty, compact, and convex set, and $f : A \rightrightarrows A$ be an upper hemicontinuous correspondence such that $f(x) \subseteq A$ is non-empty and convex for each $x \in A$. Then, f has a fixed point, i.e., there exists $x \in A$ such that $x \in f(x)$.*

The existence result is the following.

Proposition 10.3. *If the aggregate excess demand function $z(p)$ satisfies the properties in Proposition 10.2, then there exists a price vector p such that $z(p) = 0$. Therefore, in such economy a Walrasian equilibrium exists.*

Proof. Check Mas-Colell et al. (1995, p. 586). □

Things to read. This lecture is based on Mas-Colell et al. (1995, pp. 578-587).

10.1 Exercises

Exercise 10.1. Show that if demand equals supply in $k-1$ markets, then it also equals supply in the k-th market. (Hint: Use Walras' law.)

Solution to Exercise 10.1. Let $z(p) = (z_1(p), z_2(p), \ldots, z_k(p))$ be the aggregate excess demand function, where $z_\ell(p) = D_\ell(p) - S_\ell(p)$ for each good ℓ. Suppose that for prices p, we have $z_\ell(p) = 0$ for all $\ell = 1, 2, \ldots, k-1$. This means that demand equals supply in the first $k-1$ markets.

By Walras' law, we know that

$$p \cdot z(p) = \sum_{\ell=1}^{k} p_\ell z_\ell(p) = 0.$$

Since $z_\ell(p) = 0$ for $\ell = 1, 2, \ldots, k-1$, the above equation simplifies to

$$p_k z_k(p) = 0.$$

Given that prices are strictly positive, $p_k > 0$, it follows that $z_k(p) = 0$. Therefore, demand equals supply in the k-th market as well. □

Exercise 10.2. Prove Proposition 10.1.

Solution to Exercise 10.2. ($\Rightarrow$) Suppose (x^*, p^*) is a Walrasian equilibrium. By definition, for each individual i, we have $x_i^* = D_i(p^*, e_i)$. Therefore, the excess demand for individual i at prices p^* is

$$z_i(p^*, e_i) = D_i(p^*, e_i) - e_i = x_i^* - e_i.$$

Summing over all individuals, we get

$$z(p^*) = \sum_i z_i(p^*, e_i) = \sum_i (x_i^* - e_i) = \sum_i x_i^* - \sum_i e_i.$$

Since the total allocation equals the total endowment in equilibrium, we have $\sum_i x_i^* = \sum_i e_i$. Thus,

$$z(p^*) = 0.$$

($\Leftarrow$) Conversely, suppose there exists a price vector p^* such that $z(p^*) = 0$. This implies

$$\sum_i z_i(p^*, e_i) = 0.$$

Therefore,

$$\sum_i (D_i(p^*, e_i) - e_i) = 0,$$

which leads to

$$\sum_i D_i(p^*, e_i) = \sum_i e_i.$$

This means that the total demand equals the total endowment at prices p^*. For each individual i, let $x_i^* = D_i(p^*, e_i)$. Then, the allocation $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ satisfies the market-clearing condition. Since each individual chooses their most preferred bundle in their budget set at prices p^*, the pair (x^*, p^*) constitutes a Walrasian equilibrium. $\square$

Exercise 10.3. Prove property 1. of the excess demand function in Proposition 10.2.

Solution to Exercise 10.3. To prove that the excess demand function $z(p)$ is homogeneous of degree zero, we need to show that for any positive scalar $\alpha > 0$,

$$z(\alpha p) = z(p).$$

By definition, the excess demand function for individual i is given by

$$z_i(p, e_i) = D_i(p, e_i) - e_i,$$

where $D_i(p, e_i)$ is the Walrasian demand function.

The Walrasian demand function $D_i(p, e_i)$ is homogeneous of degree zero in prices. This means that for any positive scalar $\alpha > 0$,

$$D_i(\alpha p, e_i) = D_i(p, e_i).$$

Using this property, we can compute the excess demand at prices αp:

$$z_i(\alpha p, e_i) = D_i(\alpha p, e_i) - e_i = D_i(p, e_i) - e_i = z_i(p, e_i).$$

Summing over all individuals, we have

$$z(\alpha p) = \sum_i z_i(\alpha p, e_i) = \sum_i z_i(p, e_i) = z(p).$$

Therefore, we conclude that

$$z(\alpha p) = z(p),$$

which shows that the excess demand function $z(p)$ is homogeneous of degree zero. □

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