A Solarian Malthus Model

This exercise is set in the world of Solaria, a fictional planet in the Novel "The Naked Sun" by Isaac Asimov. In Solaria few people live without much interaction with each other. The birth rate is completely controlled by the government, as newborns are bred in vitro. Each inhabitant owns a massive amount of land and the space per capita is quite high. Both the land and the house of citizens are completely administered by robots so that the product per capita is huge without the need for a high increase in population. If you want to know more, check the links I provided or read the book! This exercise is a study of the economy of Solaria via the lens of the Malthus Model seen in class.

Assume there are two types of inhabitants in Solaria. The first kind likes robots a lot, while the second a little bit less. The utility function of a generic citizen i is $u_i\left(R_i\right)=\gamma_i\cdot log\left(R_i\right)-cR_i$, where γ_i measures taste for robots and c is the cost of buying a new one. People who like robots a lot are indexed with $i=h$ and they have $\gamma_i=\gamma_h$. Instead, does who do not like robots as much are $i=\ell$ and have $\gamma_i=\gamma_\ell<\gamma_h$.

a. What is the optimal amount of robots for each citizen i? Interpret your result.

Each citizen chooses the amount of robots that maximise his utility function. The problem is the following.

$$\underset{R_i\geq 0}{max}\ \ u_i\left(R_i\right)=\gamma_i\cdot log\left(R_i\right)-cR_i$$

As usual, we take the derivative and set it equal to 0 (what about second order conditions?).

$$\begin{aligned}\frac{\partial\ u_i\left(R_i\right)}{\partial\ R_i}&=\frac{\gamma_i}{R_i}-c=0\\ &\Rightarrow R_i^*=\frac{\gamma_i}{c}\end{aligned}$$

The interpretation is quite straightforward. The higher the tastes for robots γ_i, the higher the amount of robots. On the other hands, if the cost c increases then citizen i will buy less robots.

b. A fraction λ of the population likes robots a lot, while the rest $(1-\lambda)$ not that much. If the level of population is P, how many robots R are there in total? How do you interpret $\overline{\lambda} = \lambda\gamma_h + (1-\lambda)\,\gamma_\ell$?

The share λ of the population will buy an amount $R_h^* = \frac{\gamma_h}{c}$ of robots, while $(1-\lambda)$ will get $R_\ell^* = \frac{\gamma_\ell}{c}$. If the population is P, then the total amount of robots will be

$$\begin{aligned} R &= \lambda P \frac{\gamma_h}{c} + (1-\lambda)\, P \frac{\gamma_\ell}{c} \\ &= \frac{P}{c} \left(\lambda\gamma_h + (1-\lambda)\,\gamma_\ell\right) \\ &= \frac{P}{c}\overline{\lambda} \end{aligned}$$

The variable $\overline{\lambda}$ captures the average tastes for robots in the population. The higher the λ, γ_h or γ_ℓ, the higher the citizens of Solaria like robots on average.

c. Production is given by the number of robots in the economy. Assume that production is linear in robots $Y(R) = \alpha_y + \beta_y R$. Write it as a function of population explicitly. What is production per capita y?

This is just a matter of substitution.

$$\begin{aligned} Y(R) &= \alpha_y + \beta_y R \\ &= \alpha_y + \beta_y \frac{P}{c} \left(\lambda\gamma_h + (1-\lambda)\,\gamma_\ell\right) \\ Y(P) &= \alpha_y + \beta_y \frac{P}{c}\overline{\lambda} \end{aligned}$$

As usual, production per capita is $y = \frac{Y}{P}$.

$$y(P) = \frac{Y(P)}{P} = \frac{\alpha_y}{P} + \beta_y \frac{\overline{\lambda}}{c}$$

This is the first step for constructing a Malthus model from scratch, we miss the law of motion of population, which is the state variable.

d. I mentioned that the birth rate is controlled, how do you translate this assumption? What is $b(y)$?

Since the amount of newborns is fully controlled by the government and does not depend on production, it is equal to a constant $b\left(\cancel{y}\right) = k$ where k is a constant.

<u>*Question:*</u> Do you have any ideas about real circumstances in which the same is true?

e. From now on assume that $c = 1$. The mortality rate is linear in production per capita $m(y) = \alpha - \beta y$. You have all the ingredients. Find the steady-state level of production per capita and population. Represent everything in a nice graph.

As usual, the law of motion of population is $\dot{P}_t = \left[b - m\left(y\right)\right] P_t$. Hence, in steady-state we need $\dot{P} = 0$, which means $P = 0$ or $b = m(y)$. The second condition implies the following.

$$\begin{aligned} k &= \alpha - \beta y \\ \Rightarrow \beta y &= \alpha - k \\ \Rightarrow y^* &= \frac{\alpha - k}{\beta} \end{aligned}$$

As for population, we can find its steady-state level by exploting the relation between P and y. By substituting y^*we obtain:

$$\begin{aligned} y^* &= \frac{\alpha_y}{P*} + \beta_y \overline{\lambda} \\ \frac{\alpha - k}{\beta} &= \frac{\alpha_y}{P*} + \beta_y \overline{\lambda} \\ \frac{\alpha - k}{\beta} - \beta_y \overline{\lambda} &= \frac{\alpha_y}{P*} \\ P^* \left[\frac{\alpha - k}{\beta} - \beta_y \overline{\lambda}\right] &= \alpha_y \\ P^* \left[\frac{\alpha - k - \beta_y \beta \overline{\lambda}}{\beta}\right] &= \alpha_y \\ P^* &= \frac{\alpha_y \beta}{\alpha - k - \beta_y \beta \overline{\lambda}} \end{aligned}$$

To draw the curve of P as a function of y

we employ their relation found previously.

$$
\begin{aligned}
y(P) &= \frac{\alpha_y}{P} + \beta_y \overline{\lambda} \\
y - \beta_y \overline{\lambda} &= \frac{\alpha_y}{P} \\
P \left[y - \beta_y \overline{\lambda} \right] &= \alpha_y \\
P &= \frac{\alpha_y}{y - \beta_y \overline{\lambda}}
\end{aligned}
$$

We can now put the model to use and see if we can learn something about Solaria.

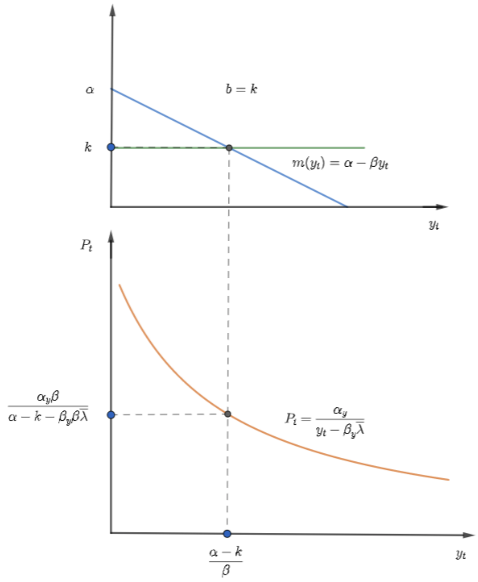

Figure 1: Steady-state level of population and production per capita.

f. How does the steady-state level of population depends on $\overline{\lambda}$ or, λ and γ_h, γ_ℓ? Try to simulate a positive shock to $\overline{\lambda}$ and see what happens in the steady state.

The only curve that is affected by an increase in $\overline{\lambda}$ is P as a function of y. The following picture represents the change. In the new steady state y is unaffected, while P increases. All the production of the increased number of robots is eaten by new population, and resources per capita stay the same. Is this what is going on in Solaria? Maybe we have to look at something else.

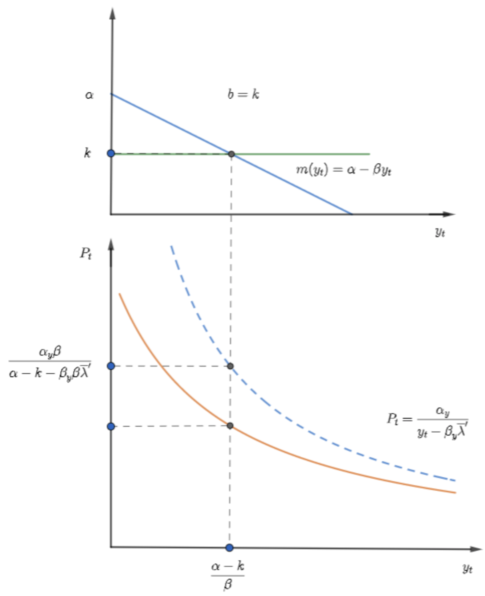

Figure 2: Positive shock to $\overline{\lambda}$.

g. What happens if the government decides to lower the birth rate? What do you think the birth rate and $\overline{\lambda}$ are in Solaria? Any comments?

The only curve affected by a decrease in k is the birth rate. The new situation is depicted in the figure here. The lower number of births leads to a decrease in population, and hence those who remains enjoy more resources per capita. This is more or less what happens in Solaria, so maybe this channel is the relevant one to study its economy. Or maybe this is the wrong model to tackle the problem 🙂. What do you think?

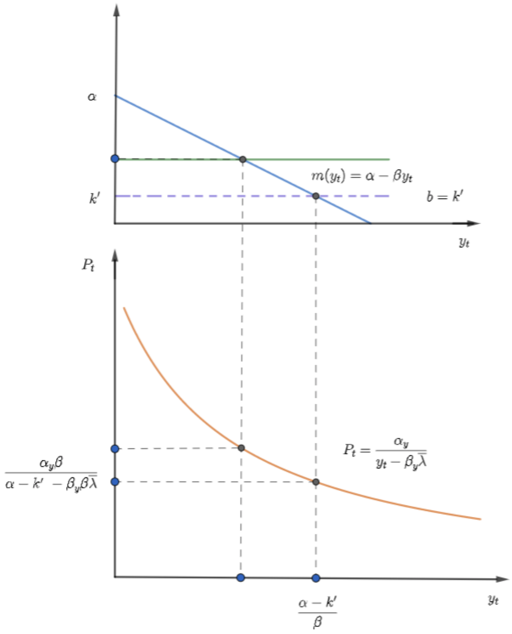

Figure 3: Negative shock to k.