Baby boom (from midterm 2019)

This exercise is not significantly different to exercises 2 and 3 of TD1! You should already know how to do the computations and the graph of the first questions. The second part is a little bit more difficult and it requires you to translate what is happening in mathematics. We have $F(k_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$. Investment is, as always, $I = sY = sF(K_t, L_t)$. The law of motion of capital is $\Delta K_t = K_{t+1} - K_t = I - \delta K_t$. The work population grows at rate $n = \frac{L_{t+1} - L_t}{L_t} = 0.05$. We also have that $\delta = 0.05,\ s = 0.1$ and $\alpha = 0.5$.

Remark: In these solutions I spell out every step just for you to understand, as you can see from the professor's solutions it is not needed that you specify each step as I do. However, for sure it can not hurt you.

a. On the balanced growth path, k and y are stable. Compute their numerical values.

The answer to this question requires no more than performing the computations we have been doing in the previous TDs and substituting numbers. First, we have to convert everything in per capita terms. The production function becomes:

$$
\begin{aligned}
\frac{1}{L_t} F(K_t, L_t) &= F\left(\frac{K_t}{L_t}, \frac{L_t}{L_t}\right) \\
&= \left(\frac{K_t}{L_t}\right)^{\alpha} \left(\frac{L_t}{L_t}\right)^{1-\alpha} \\
f(k_t) &= k_t^{\alpha}
\end{aligned}
$$

You should recognise that we have been using this function a lot! Is the classical Cobb-Douglas. To compute the numerical value of k in the balanced growth path we can rely on the condition under which this variable is indeed on such path, which means that its growth rate is equal to zero. Hence, we must first compute its growth rate.

$$
\begin{aligned}
\frac{\Delta k_t}{k_t} &= \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} \\
&= \frac{sF(K_t, L_t) - \delta K_t}{K_t} - n \\
&= \frac{s\frac{1}{L_t}F(K_t, L_t) - \delta \frac{1}{L_t}K_t}{\frac{1}{L_t}K_t} - n \\
\frac{\Delta k_t}{k_t} &= \frac{sf(k_t) - \delta k_t}{k_t} - n \\
\Delta k_t &= sf(k_t) - \delta k_t - nk_t \\
&= sf(k_t) - (\delta + n)k_t
\end{aligned}
$$

The condition for being in steady state its growth rate equal to 0, thus, we check what this condition implies in this exercise.

$$\Delta k_t = 0 \Leftrightarrow sf(k_t^*) - (\delta + n)k_t^* = 0 \Leftrightarrow s(k^*)^\alpha = (\delta + n)k_t^*$$

By substituting the numbers we are given in the text we obtain that:

$$k^* = \left(\frac{\delta + n}{s}\right)^{\frac{1}{\alpha - 1}} = \left(\frac{0.05 + 0.05}{0.1}\right)^{-2} = 1$$

Since $y^* = f(k^*)$ we obtain:

$$y^* = (k^*)^\alpha = 1^{1/2} = 1$$

b. Is the savings rate s is at its golden rule value? If not, what should be the golden-rule savings rate?

From problem 2 of TD 1 you may recall that the golden rule savings rate is $s = \alpha = 0.5$, while in this case we have that $s = 0.1 \neq 0.5$! However, let's try to prove it again. There are many ways to do it. The first one is to express consumption as $c^* = (1 - s)f(k^*)$ and compute the s that maximises it in steady state. The steps to do this are detailed in the solution of TD 1. A different method could be to check the k which comes from the golden rule of savings and derive the s for which we get that k. The steps are the following. First, express consumption only as a function of capital:

$$
\begin{aligned}
c^* &= (1-s)y^* \\
&= (1-s)f(k^*) \\
&= f(k^*) - sf(k^*) \\
&= f(k^*) - (\delta+n)k^*
\end{aligned}
$$

Where the last step come from the steady state condition we derived in the previous point. We then obtain the k that maximises consumption by solving a maximisation problem:

$$
\begin{aligned}
\frac{\partial c^*}{\partial k^*} = 0 &\Rightarrow f'(k^*) - (\delta+n) = 0 \\
&\Rightarrow \alpha(k^*)^{\alpha-1} = (\delta+n) \\
&\Rightarrow k^* = \left(\frac{\delta+n}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
&\Rightarrow k^* = \left(\frac{0.05+0.05}{0.5}\right)^{\frac{1}{0.5-1}} = 25
\end{aligned}
$$

Which is quite different from the $k^* = 1$ we got before! What is the s that rationalises this result? From the expression we derived in the previous point we have:

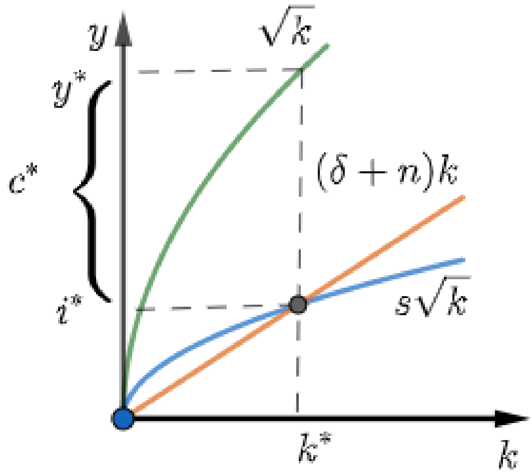
$$
\begin{aligned}
k^* &= \left(\frac{\delta+n}{s}\right)^{\frac{1}{\alpha-1}} \\
25 &= \left(\frac{0.05+0.05}{s}\right)^{\frac{1}{0.5-1}} \\
25 &= \left(\frac{0.1}{s}\right)^{-2} \\
25 &= \left(\frac{s}{0.1}\right)^{2} \\
5 &= \left(\frac{s}{0.1}\right) \\
0.5 &= s \neq 0.1
\end{aligned}
$$

Which indeed gives us $\alpha = s$. Remember that $i^* = sy^* = 0.1(1) = 0.1$, moreover $c^* = y^* - i^* = 1 - 0.1 = 0.9$.

c. Make a plot like the ones made in class with k on the horizontal axis and y on the vertical axis. Draw $f(k) = k^{0.5}$. Draw $sf(k)$. Draw $(n + \delta)k$. Indicate k^* and y^*.

The graph is ecxactly the one we did in exrcise 3 odf TD 1! I'll report the same picutre I put there. The production function is $f(k_t) = (k_t)^{\frac{1}{2}} = \sqrt{k_t}$, while $\delta = n = 0.05$, $y^* = 1$ and $k^* = 1$ (you may want to substitute the numbers in your graph).

1. $\sqrt{k}$
2. $(\delta + n)k_t$
3. $s\sqrt{k}$



Graph from exercise 3 TD1.

d. After World War 2, many American soldiers fighting in Europe came back home and made lots of babies. Imagine that during World War 2, the American economy is on the balanced growth path. Then when troops come home at time t, the population L doubles. What is the numerical value of the growth rate of k_t just after the doubling? (If you don't have a calculator, you can leave a mathematical expression as is (as long as it just involves numbers, no variables).)

We are asked to compute the growth rate of k_t at time t, when the populatoin doubles. We have to perform the exact computations we did in the previous points, but instead of having L_t, we have $2L_t$. We start from the production function:

$$
\begin{aligned}
\frac{1}{2L_t}F(K_t, 2L_t) &= F\left(\frac{K_t}{2L_t}, \frac{2L_t}{2L_t}\right) \\
&= \left(\frac{K_t}{2L_t}\right)^{\alpha}\left(\frac{2L_t}{2L_t}\right)^{1-\alpha} \\
f(k_t) &= \left(\frac{k_t}{2}\right)^{\alpha} = \bar{k}_t^{\alpha}
\end{aligned}
$$

Now we have to compute the growth rate. Again the calculations follow the same logic as before:

$$
\begin{aligned}
\frac{\Delta k_t}{\frac{k_t}{2}} &= \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} \\
&= \frac{sF(K_t, 2L_t) - \delta K_t}{K_t} - n \\
&= \frac{s\frac{1}{2L_t}F(K_t, 2L_t) - \delta\frac{1}{2L_t}K_t}{\frac{1}{2L_t}K_t} - n \\
\frac{\Delta k_t}{\frac{k_t}{2}} &= \frac{sf(k_t) - \delta\frac{k_t}{2}}{\frac{k_t}{2}} - n \\
\Delta k_t &= sf(k_t) - \delta\frac{k_t}{2} - n\frac{k_t}{2} \\
&= sf(k_t) - (\delta + n)\frac{k_t}{2}
\end{aligned}
$$

By expliciting the production function we obtain:

$$
\Delta k_t = s\left(\frac{k_t}{2}\right)^{\alpha} - (\delta + n)\frac{k_t}{2} = 0.1\left(\frac{1}{2}\right)^{0.5} - (0.05 + 0.05)\frac{1}{2} = 0.0207
$$

However, we must find the growth rate, not only the Δ:

$$
\frac{\Delta k_t}{\frac{k_t}{2}} = \frac{0.0207}{\frac{1}{2}} = 0.0414
$$

e. As previously, imagine that during World War 2, the American economy is on the balanced growth path. Then when troops come home at time t, the population L doubles. Also, since these young people want to make babies as fast as possible, at the

same moment, the growth rate jumps from $n = 0.05$ to $n = 0.15$. Reproduce the graph in c. and show how the long run equilibrium will change. Indicate what happens to k just after soldiers get back (at time t) and where the economy converges in the long run (indicate numerical values).

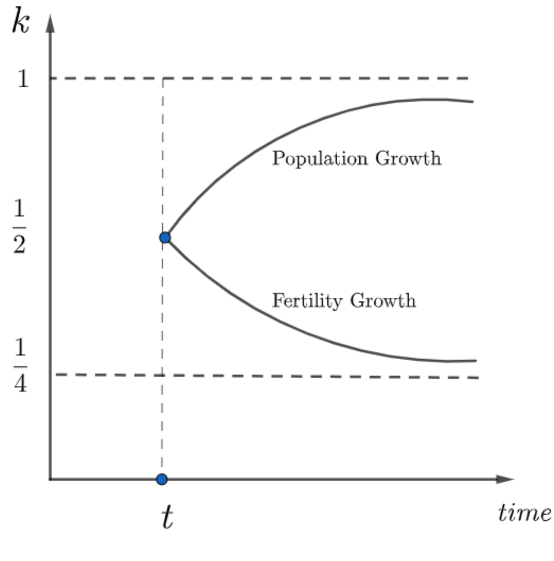
Before showing the graph we need to compute the new k^* and y^*. As always, we set the law of motion equal to zero:

$$\begin{aligned}
\Delta k_t &= 0 \\
&\Leftrightarrow s\left(\frac{k_t^*}{2}\right)^{\alpha} - (\delta + n)\frac{k_t^*}{2} = 0 \\
&\Leftrightarrow \left(\frac{k_t^*}{2}\right)^{\alpha-1} = \frac{\delta + n}{s} \\
&\Leftrightarrow \left(\frac{k_t^*}{2}\right) = \left(\frac{0.05 + 0.015}{0.1}\right)^{\frac{1}{0.5-1}} = \frac{1}{4}
\end{aligned}$$

As for y^* we have that $y^* = f(k^*) = \left(\frac{k_t^*}{2}\right)^{0.5} = \left(\frac{1}{4}\right)^{0.5} = \frac{1}{2}$. I do not include the graph here again as it is the same graph as before with different numbers.

f. Plot k over time around the period t (hence make a graph with time on the horizontal axis and k on the vertical axis). On the same graph, show how k adjusts after t in the situation described in d. and in the situation described in e. Make sure to show where k is converging in each case.

At time t the input in the production function is immediately halved due to the increase in population. If there is no fertility growth after a while k will return on its original balanced growth path. In the case of fertility growth, instead, it will converge to $\frac{1}{4}$.



g. In the context of this model, are American workers (those always in the US) better off before time t (before the influx of workers), some time after time t if fertility does not change d. some time after t or if fertility does change e.? Rank the three situations from best to worst and justify briefly.

This question just amounts to compare the y^* in different circumstances. In $\tau < t$ we have $y_\tau^* = 1$, when fertility increase $(n = 0.15)$ occurs at time t we have that the new steady state gdp is $y^* = \frac{1}{2}$, while without the increase in n but after the increase in population we are outside the growth path leading to $y^* = 1$, and therefore we are slightly below this value. Therefore, the best situation is the one before the shock, then we have the increase in population without the increase in n and lastly the worse situation is when we also observe a fertility rate increase.