Midterm 2020

Remark: Some solutions here are way more elaborate then what was needed to get full points! This is just to make you understand.

2 - Production in the Solow-Swan model

Here we have a Solow-Swan model with population growth $n \in (0,1)$, no technological growth, where $s \in (0,1)$ is the saving rate, K is the stock of capital, $k = \frac{K}{L}$ is capital per worker, $F(K,L)$ is the production function with the usual assumptions, $f(k) = \frac{F(K,L)}{L}$ is production per worker, $\delta \in (0,1)$ is the rate of capital depreciation, and finally k^* denotes k on the balanced growth path.

In this exercise I put both the general answer and the particular case in which $f(k) = k^\alpha$ to show you that the solutions make sense and to make you visualise them better.

Is $s\frac{\delta f(k)}{\delta k} - \delta - n$ greater than zero, equal to zero or smaller than zero or we cannot say without more information?

▼ **1.** *If $k \to 0$*. In this case $\frac{\delta f(k)}{\delta k} \to \infty$ due to the Inada condition, therefore the left side of the equation is way bigger then the right side and therefore the expression is positive.

Example: In the special Cobb-Douglas case $\frac{\delta k^\alpha}{\delta k} = \alpha k^{\alpha-1} = \frac{\alpha}{k^{1-\alpha}}$. Since k is at the denominator you can clearly see that if it goes to 0 then the ratio goes to infinity.

▼ **2.** *If $k = k^*$*. A lot of students got this wrong. The standard logic I saw was that since $k = k^*$ we are on the balanced growth path and therefore $sf(k^*) = (\delta - n)k^*$, hence $sf(k^*)' = (\delta + n)$. However, this logic is fallacious. If two functions are equal in one point, which in our case is k^*, then it is not true that their derivative is equal in that point in general. In fact, it is true only if those two functions are equal in every point. This is kind of a technical argument, I understand it is not straightforward, but maybe you will be convinced by the solution. Remember that the derivative of a function tells us how much that function varies after an infinitesimal change in the variable you take the derivative with respect to. Moreover, you know two things. For k a little bit smaller than k^*, say k_- we have that $sf(k_-) > (\delta + n)k_-$ (you can see it from the standard graph). Also, if k is a little bit higher than k^*, call it k_+ it holds that $sf(k_+) < (\delta + n)k_+$. This means that in the process of going from k_- to k_+ the function $(\delta + n)k$ had a higher increase then $sf(k)$, and therefore its derivative $(\delta + n)$ is higher then the derivative $sf'(k)$. Since this holds for all $k_- < k^* < k_+$ then it also holds for k^*.

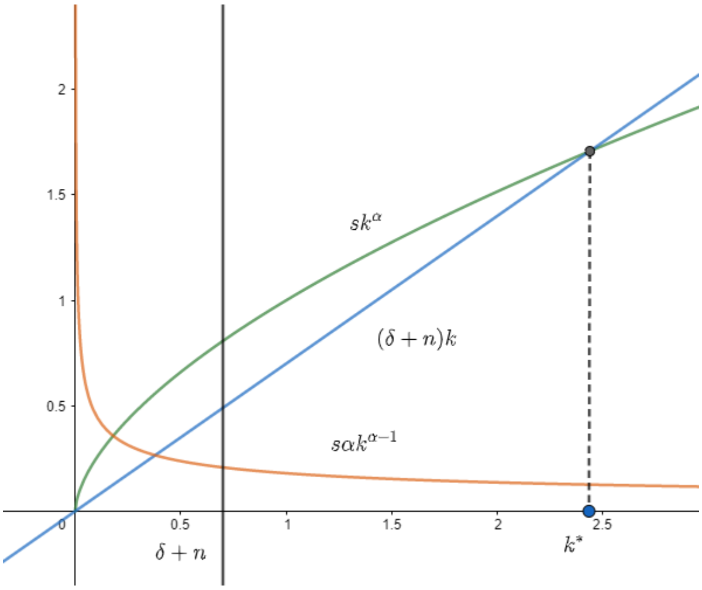

Figure 1: Graph with $f(k) = k^\alpha$, $s = 0.3$, $\alpha = 0.6$, $\delta = 0.3$, $n = 0.4$.

▼ **3.** *If $k \to \infty$*. This is the converse of the point we had before. Due to the Inada conditions $\frac{\delta f(k)}{\delta k} \to 0$ and therefore only the right (negative) side of the equation is left.

<u>*Example*</u>: As before, you can check that $\frac{\alpha}{k^{1-\alpha}} \to 0$ and therefore the general reasoning holds also here.

▼ **4.** *Is $\frac{\delta f(k)}{\delta k} - \delta - n$ greater than zero, equal to zero or smaller than zero or we cannot say without more information if $k = k^*$? (Note the difference is that here there is no s in the expression.).* I saw two different kind of mistakes in this question. The first one was to assume that we were in the golden rule and therefore $\frac{\delta f(k^*)}{\delta k} = \delta - n$. However, this was not specified by the question. We could have any $s \in (0, 1)$. The other typical mistake was to say that since $s\frac{\delta f(k^*)}{\delta k} = \delta - n$ and $s < 1$ then if you remove it you get $\frac{\delta f(k^*)}{\delta k} > \delta - n$. This logic is faulty, as it does not consider that $\frac{\delta f(k^*)}{\delta k}$ itself depends on s, and therefore it is different for different values of s. Hence, we can not determine the relationship between the two quantities.

<u>*Example*</u>: In the Cobb-Douglas case we have that $k^* = \left(\frac{\delta+n}{s}\right)^{\frac{1}{1-\alpha}}$, therefore $f'(k^*) = \alpha\left(\frac{\delta+n}{s}\right)^{\frac{1-\alpha}{1-\alpha}} = \alpha\left(\frac{\delta+n}{s}\right)$, therefore the question becomes: what is the sign of $\alpha\left(\frac{\delta+n}{s}\right) - \delta - n$? As an example:

$$\alpha\left(\frac{\delta+n}{s}\right) - \delta - n > 0$$
$$\alpha\left(\frac{\delta+n}{s}\right) > \delta + n$$
$$\frac{\alpha}{s} > 1$$
$$\alpha > s$$

3 - A general purpose technology

Consider the production function with general purpose technology $Y = F(A, K, L) = AK^{\alpha}L^{1-\alpha}$ where $\alpha \in (0, 1)$, K is the capital stock and L is the labor. Assume that $A_{t+1} = (1 + g)A_t$ and $L_{t+1} = (1 + n)L_t$ where g and n are exogenous constants. Capital depreciates at rate δ. You can use the same approximation as in the TD for the growth of some variable X: $g_X = log\left(\frac{X_{t+1}}{X_t}\right)$.

This exercise consists in playing with growth rates, the techniques are the same we used in the TDs.

▼ **1.** *If the growth rate of capital at time t is $g_{K,t}$, compute the growth rate of Y at time t in terms of $g_{K,t}$ and the growth rates of technology and labour.* Most if you got this question right. The idea is to exploit the definition given in the text and do the calculations we did in the TDs. Some of you thought that the α exponent was also on A, probably it was just a misreading of the text. Also, some students took derivatives with respect to time, but here time is discrete!

$$\begin{aligned}
g_{Y,t} &\approx log\left(\frac{Y_{t+1}}{Y_t}\right) \\
&= log\left(\frac{A_{t+1}K_{t+1}^{\alpha}L_{t+1}^{1-\alpha}}{A_tK_t^{\alpha}L_t^{1-\alpha}}\right) \\
&= log\left(\frac{A_{t+1}}{A_t}\right) + \alpha log\left(\frac{K_{t+1}}{K_t}\right) + (1-\alpha)log\left(\frac{L_{t+1}}{L_t}\right) \\
&= g + \alpha g_{K,t} + (1-\alpha)n
\end{aligned}$$

▼ **2.** *Let's now define general-purpose technology-adjusted labor as $\bar{L} = A^{\frac{1}{1-\alpha}}L$. What is the growth rate of $\bar{L}$?* The procedure for answering this question is the same as before, we just need to use the expression of $\bar{L}$. Most of you got this correct.

$$\begin{aligned}
g_{\bar{L}} &= log\left(\frac{A_{t+1}^{\frac{1}{1-\alpha}}L_{t+1}}{A_t^{\frac{1}{1-\alpha}}L_t}\right) \\
&= \frac{1}{1-\alpha}log\left(\frac{A_{t+1}}{A_t}\right) + log\left(\frac{L_{t+1}}{L_t}\right) \\
&= \frac{1}{1-\alpha}g + n
\end{aligned}$$

▼ **3.** *What will be the growth of output per worker in the long run?* As always, output per worker is $\frac{Y}{L}$, some of you got confused by the $\bar{L}$ and computed the wrong growth rate. So we have (from now on I omit the logarythm transformation):

$$\frac{Y}{L} = \frac{AK^{\alpha}L^{1-\alpha}}{L} = A\left(\frac{K}{L}\right)^{\alpha}$$

$$g_y = g + \alpha(g_{K,t} - n)$$

▼ **4.** *What will be the growth rate of real wages in the long run?* Real wage is nominal wage times units of technology adjusted labour *over number of workers*. Before starting, notice that First, notice that $g_{\bar{A}} = \frac{g}{1-\alpha}$. We can now compute the growth rate:

$$w_r = \frac{wLA^{\frac{1}{1-\alpha}}}{L} = wA^{\frac{1}{1-\alpha}}$$

$$g_{w_r} = g_w + g_{\bar{A}} = \frac{g}{1-\alpha}$$

▼ **5.** *What will be the growth rate of the real interest rate in the long run?* From the lecture notes you know that r is stable, which implies that it's growth rate is 0 (not generally constant!).

5 - Steady-States

In this exercise you had to deal with a strange production function. We briefly talked about it in one of the TDs, but if you did not remember you had to think carefully to get the answer correct.

▼ **1.** *In the Solow-Swan model with constant population and technology, how many steady-states are there in total if $\frac{\delta F(K)}{dK} = c$ where c is a constant?* Here it was key to understand that the production function is linear. Consider $F(K) = a + cK$, then you have $\frac{\partial F(K)}{\partial K} = c$. We already saw what happens if the production function is linear in TD2. First, there is a steady state in 0. Second, there could be other infinitely many steady states if $\delta = sc$, which is in fact the condition for being in a steady state. The answer to this question should be clear from the graph below.

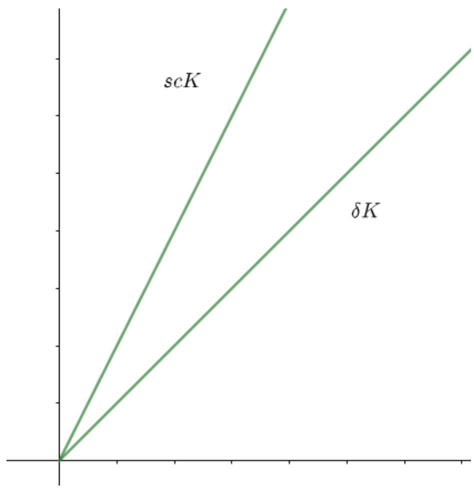

Figure 2: Steady state with linear production function.

▼ **2. (Bonus)** *In the Solow-Swan model with constant population and technology, how many stable steady-states are there in total if $\frac{\delta F(K)}{dK} = c$ where c is a constant? A steady-state is stable if for small deviation around K, the economy returns to K automatically.* This question was a bonus as you did not work with the "stability" concept before. However, is exactly what we did in the previous TD, so you should be a little bit familiar with it by now. Consider the two different cases we studied before. First, it could be that $sc \neq \delta$ and therefore we only have one steady state at 0. Is it stable? What happens if we perturb it a little bit and move on $0 + \epsilon$? It depends. If $sc > \delta$ then there is more investment then depreciation, and K increases more than what is lost due to δ. In this case the steady state is not stable, as we do not return to 0. If instead $sc < \delta$, then after a small increase capital still depreciates at a higher rate then what is saved. Hence, we return back to 0 and the state is stable. In the case $sc = \delta$ any point is a steady state and therefore no state is stable, as if we move a little bit we are already in a new steady state.

6 - An Algal Bloom

Consider the dynamics of the fish population with fishing. At time $t-1$, the stock of fishes is at its long term equilibrium $S^* > 0$. Then, at time t, an algal bloom kills half of the fish population. The bloom lasts a single period. At $t+1$, the algal bloom has ended and the ecosystem is back to where it was before (except that fishes have died of course).

This exercise did not require hard computations, you had to reason about the question and understand more or less intuitively the direction of the answer.

▼ **1.** *At $t+1$, is the net growth of the stock $\Delta S_{t+1} = S_{t+2} - S_{t+1}$ higher, equal or lower than the net growth at $t-2$, or we don't have enough information to tell. Assume that the number of boats have not changed between $t-2$ and $t+1$.* The question here asks to compare ΔS_{t+1} to $\Delta S_{t-2} = S_{t-1} - S_{t-2}$. Since before t we were in steady state, we must have that $\Delta S_{t-2} = 0$ and $S_{t-1} = S_{t-2} = S^*$. Hence, the question becomes: is $\Delta S_{t+1} > 0$? At time t we have the algal bloom, so the steady state population gets halved and we have that $S_t = \frac{S^*}{2}$. At $t+1$, even if the bloom is over, we are not at the steady state as $\frac{S^*}{2} < S^*$. To reach the steady state again the stock of fish must grow positively. For some time periods τ, we will have that $S_{\tau+1} > S_\tau$. We reached the conclusion that $\Delta S_{t+1} > 0$. Not all of you got this correct, I think it may be due to the confusion with the definition of net growth rate.

▼ **2.** *At $t+1$, is the natural growth of the stock $\tau(S_{t+1})$ higher, equal or lower than the natural growth at $t-2$ $\tau(S_{t-2})$, or we don't have enough information to tell. Assume that the number of boats have not changed between $t-2$ and $t+1$.* This question asks to compare the levels of the parabola for different values of S_t. In S_{t-2} we were in steady state, but we do not know where! As you can see from the graph, values of $\tau\left(\frac{S^*}{2}\right)$ could be both higher or lower than $\tau(S^*)$.

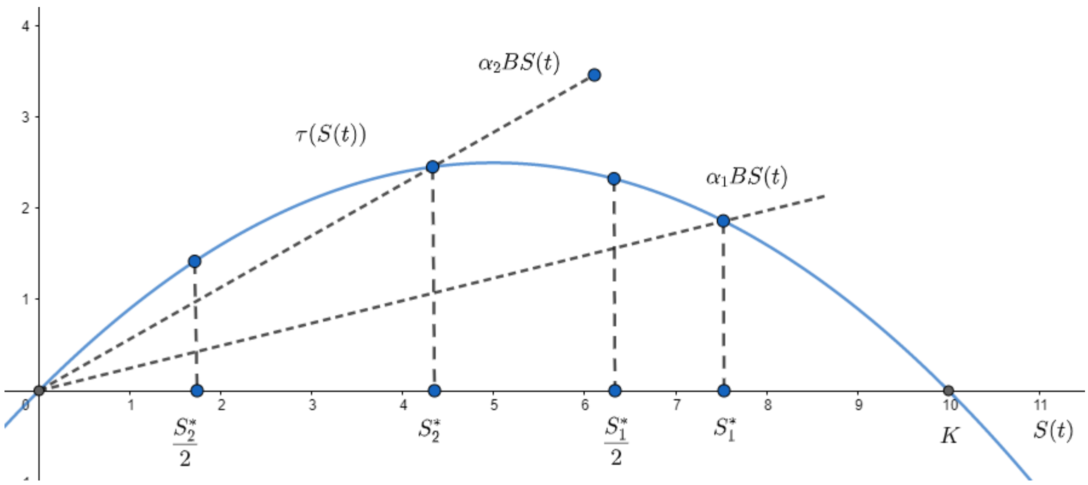

Figure 3: Steady states and growth rate for different fishing intensities.

▼ **3.** *If commercial fishing was very intense, could this bloom cause the population to go extinct?* Here you should consider that fishermans' technology is $\alpha < 1$, therefore they will always be able to catch a fraction of the fish that is lower than the total amount available. If $\alpha \to 1$ there will still be a small fraction ϵ of fishes around. You can see from the graph of the growth rate τ that this is positive at any value greater than 0, so extinction is not possible in this model (does not mean it is not possible in reality!).