TD1

1 Review Questions

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a. Inequality of income across countries was already very wide at the end of the medieval period.

- *Answer*

  False: It started widening after the Industrial Revolution. Before the revolution, growth was almost null around the world. Production was very close to the subsistence level and there were no investments. Technological progress was the catalyst of growth and, since it was heterogeneous across countries, induced an increase in inequality.

2 The Golden Rule for Savings

This problem aims to get you used to the basic logic of the Solow-Swan model. This exercise is a particular case in which we have a Cobb-Douglas production function, population growth and no technology. In the notation of the class, population growth is $n > 0$, technological growth is $g = 0$ and the production function is $f(k) = k^{\alpha}$ with $0 < \alpha < 1$.

Question: Can you get what is the original production function $F(K_t, L_t)$?

Question: What is the interpretation of α? Why is it between 0 and 1?

a. Express the steady-state level of consumption c^* as a function of k^* and the exogenous parameters n, δ and α (but not s).

Question: Before starting, what is the difference between exogenous and endogenous?

First, we have to recall how to express consumption in this model. What is consumed is what is produced minus what is invested, therefore $c = y - sy$. The saving rate s is exogenous, but production is a function of capital $y = f(k)$. We are also reminded that at the steady state $sf(k^*) = (\delta + n)k^*$. This is key to expressing the steady state level of consumption, c^* as a function of the exogenous variables and k^*, but not s:

$$\begin{aligned} c^* &= y^* - sy^* \\ &= f(k^*) - sf(k^*) \\ &= f(k^*) - (\delta + n)k^* \quad \text{By substituting the steady state condition} \\ &= (k^*)^{\alpha} - (\delta + n)k^* \end{aligned}$$

Here we have c^* as a function of k^*, n, δ and α.

b. Use the result to the previous question to find the optimum level of k^* from the point of view of consumption.

This question asks to maximise consumption with respect to capital. We want to answer the question: what is the level of k^* that gives me the maximum c^*? This is a standard optimisation problem. The first order condition requires setting the first derivative of the objective function, in our case consumption in the steady state, to be equal to 0:

$$\frac{\partial c^*}{\partial k^*} = 0 \Rightarrow \alpha(k^*)^{\alpha-1} - (\delta+n) = 0 \Rightarrow k^* = \left(\frac{\delta+n}{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{1}$$

Remember that you can express this quantity differently by playing with the exponent!

Question: What about the second-order condition?

c. Express the steady-state level of consumption c^* as a function of the exogenous parameters only (n, δ, α, also including s).

We have to perform the same operation as before, but without including k^* in the expression for consumption. Therefore, we must first find k^* as a function of the exogenous variables only, to be able to substitute it in the expression for consumption. We start from the fact that $sf(k^*) = (\delta + n)k^*$:

$$\begin{aligned} sf(k^*) &= (\delta+n)k^* \\ \frac{1}{k^*}(k^*)^\alpha &= \frac{\delta+n}{s} \quad \text{By substituting } f(k^*) = (k^*)^\alpha \\ (k^*)^{-1}(k^*)^\alpha &= \frac{\delta+n}{s} \quad \text{Since } \frac{1}{(k^*)} = (k^*)^{-1} \\ (k^*)^{\alpha-1} &= \frac{\delta+n}{s} \\ (k^*)^{\frac{\alpha-1}{\alpha-1}} &= \left(\frac{\delta+n}{s}\right)^{\frac{1}{\alpha-1}} \quad \text{Multiply both exponents by } \frac{1}{1-\alpha} \end{aligned}$$

$$k^* = \left(\frac{\delta+n}{s}\right)^{\frac{1}{\alpha-1}} \tag{2}$$

We can now substitute k^* in the expression for c^*, as it is expressed only as a function of exogenous variables:

$$\begin{aligned} c^* &= (1-s)f(k^*) \\ &= (1-s)(k^*)^\alpha \\ &= (1-s)\left(\frac{\delta+n}{s}\right)^{\frac{\alpha}{\alpha-1}} \end{aligned}$$

Here we have c^* which only depends on s, n, δ and α.

d. Use the result to the previous question to find the optimum level of the savings rate s from the point of view of consumption.

Again, same story as the previous point, but instead of maximising for k^* we do it for s. First, we rewrite c^* to be able to take the derivative easily. Remember that $\left(\frac{\delta+n}{s}\right)^{-a} = \left(\frac{s}{\delta+n}\right)^{a}$. By changing the fraction we had in the previous point we obtain $\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$. We also isolate the variable with respect to which we must take the derivative (only to make the computations easier, there is no need to do this if you are comfortable with derivatives):

$$c^* = (1-s)\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} = (1-s)(s)^{\frac{\alpha}{1-\alpha}}\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

We are now ready to solve the maximisation problem (here I provided very detailed calculations, if you are comfortable with calculus you do not need to write everything as I do here). First, I recall the rules for deriving a product. In general, we have the following:

$$\frac{\partial f(x)g(x)}{\partial x} = f'(x)g(x) + f(x)g'(x)$$

In our case the two functions are $f(s) = (1-s)$ and $g(s) = s^{\frac{\alpha}{1-\alpha}}$. Everything is multiplied by the constant $\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$ which does not affect the derivative (you should remember this fact, if not, ask!). First, we compute the derivative for each of the functions that compose the product:

$$g'(s) = \frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}-1} = \frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}$$
$$f'(s) = -1$$

Then, by applying the general rule:

$$\begin{aligned}
\frac{\partial[f(s)g(s)]\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}}{\partial s} &= \frac{\partial[f(s)g(s)]}{\partial s}\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} \\
&= \left[-1s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}\right]\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}$$

By setting the derivative equal to zero, we can get rid of the constant (equivalent to dividing each side of the equality by the constant itself).

$$\left[-1s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}\right]\cancel{\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}} = 0$$

We are left with the following:

$$\begin{aligned}
\frac{\partial c^*}{\partial s} = 0 \Rightarrow & \left[-1 s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha} s^{\frac{\alpha}{1-\alpha}} \frac{1}{s}\right] = 0 \\
\Leftrightarrow & \left[-1 \cancel{s^{\frac{\alpha}{1-\alpha}}} + (1-s)\frac{\alpha}{1-\alpha} \cancel{s^{\frac{\alpha}{1-\alpha}}} \frac{1}{s}\right] = 0 \\
\Leftrightarrow & \left[-1 + (1-s)\frac{\alpha}{1-\alpha} \frac{1}{s}\right] = 0 \\
\Leftrightarrow & (1-s)\frac{\alpha}{1-\alpha}\frac{1}{s} = 1 \\
\Leftrightarrow & (1-s)\alpha = (1-\alpha)s \\
\Leftrightarrow & \alpha = s
\end{aligned}$$

The s that maximises consumption is exactly α, the exponent of the production function. This is more or less intuitive, the more your function is relatively productive, as captured by α, the more you save.

Question: Again, what about the second-order conditions for this problem?

e. Comment the results obtained to questions 2 and 4: how are they related?

Consider the expressions (1) and (2) that we computed in points **b.** and **c.**

$$k_1^* = \left(\frac{\delta + n}{\alpha}\right)^{\frac{1}{\alpha - 1}} \tag{1}$$

$$k_2^* = \left(\frac{\delta + n}{s}\right)^{\frac{1}{\alpha - 1}} \tag{2}$$

Next, consider the result $\alpha = s$ obtained in point **d.**

$$\alpha = s \tag{3}$$

By combining (1) or (2) with (3) (substituting s or α in one of the two expressions) you find that $k_1^* = k_2^*$! If $s \neq \alpha$ we would have two distinct expressions maximising consumption, which is not possible if we only have one maximum in steady state, as in this case.

Question: There is another (probably many) way to answer this question, can you get it?