TD5

1 Review Questions

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c. An improvement in extractive technology always increases fish production if fishing is socially optimal.

- *Answer*

  True: From you notes (page 9), the expression for the total harvest in the social planner solution is $H_O^* = \frac{rK}{4}\left(1 - \left(\frac{c}{p\alpha K}\right)^2\right)$. You can see that an increase in α will augment total harvest even without taking derivatives.

  Question: Don't you think the comparison between this question and the solution to point **b.** is interesting?

d. An improvement in extractive technology is always a bad thing from an environmental point of view.

- *Answer*

  True: We can check from the equilibrium expression of the stock of natural resources $S^* = \frac{c}{p\alpha}$. If α increase then the stock of natural resources decreases.

3 The Dynamics of a Fish Population with Threshold

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c. What is the natural growth of the fish population at t if $S(t) = 0$? Is it also an equilibrium?

To answer this question we just need to evaluate the growth rate in the point $S(t) = 0$.

$$N(0) = r(0 - T)(1 - 0) = -rT$$

We should have expected this result, as we know that T is a threshold for the fish to grow and $S(t) = 0 < T$. Since the computed growth rate is negative and since the stock can not go lower than 0, we conclude that $S(t) = 0$ is also a steady state. Notice that $S(t)$ is the point in which the growth rate crosses the y axis, as shown in the picture below.

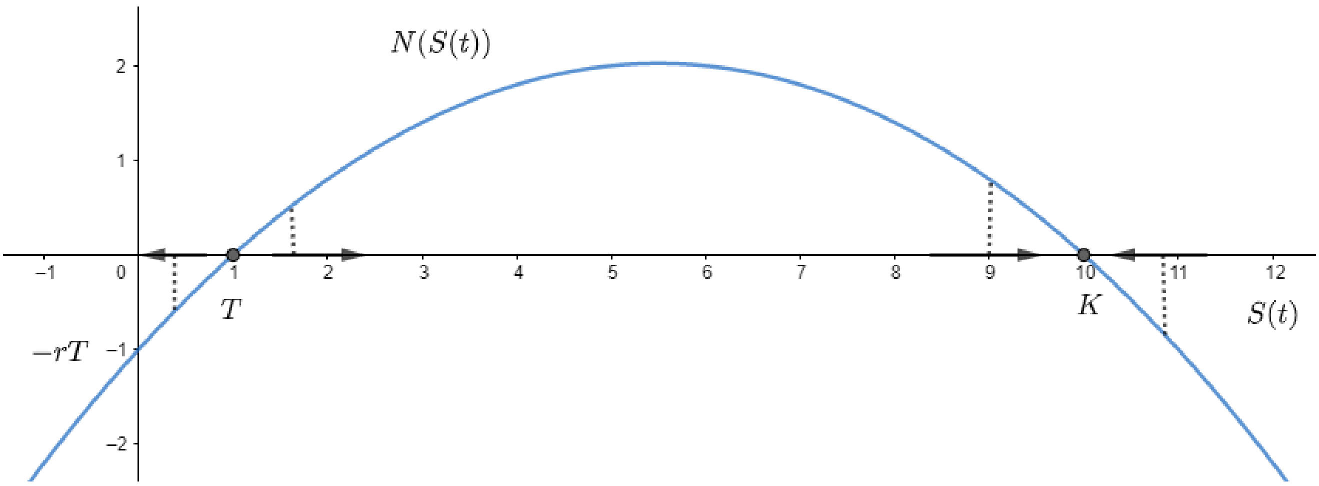

Figure 1: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$. Notice that $-rT = -1(1) = -1$, where the growth rate is negative and the stock is 0.

Question: Is this new steady state stable?

d. What is the maximum number of fish that can be caught per unit of time such that the fish population is constant? This is also called the maximum sustained yield. What is the fish stock $S(t)$ at this value?

To answer this question we must ask when the growth rate of fishes is the highest. This would allow us to capture the maximum number of fishes every time t and then obtain for $t + \varepsilon$ the greatest amount of growth so that we can always maximise our catches. Hence, we must maximise the growth rate of fishes with respect to the stock. We already have the derivative. To check for the maximum we need to find the value of $S(t)$ for which the derivative is equal to 0.

$$\begin{aligned}\frac{\partial N(S(t))}{\partial S(t)} = 0 &\Leftrightarrow r\left[\left(1 - \frac{S(t)}{K}\right) + (S(t) - T)\left(-\frac{1}{K}\right)\right] = 0\\ &\Leftrightarrow \cancel{r}\left[1 - \frac{2S(t)}{K} + \frac{T}{K}\right] = 0\\ &\Leftrightarrow K - 2S(t) + T = 0\\ &\Leftrightarrow S(t) = \frac{K+T}{2}\end{aligned}$$

Now that we have the stock of fishes that maximises growth, we can ask by how much fish grows for this value of the stock. Of course, to answer this question we just need to plug the value we just found in the growth rate.

$$\begin{aligned}N\left(\frac{K+T}{2}\right) &= r\left(\frac{K+T}{2} - T\right)\left(1 - \frac{K+T}{2K}\right)\\ &= r\left(\frac{K+T-2T}{2}\right)\left(\frac{2K-K-T}{2K}\right)\\ &= r\left(\frac{K-T}{2}\right)\left(\frac{K-T}{2K}\right)\\ &= r\frac{(K-T)^2}{4K}\end{aligned}$$

This expression tells us by how much fish we can catch for growth to always be at its maximum.

e. Graph the fish growth function $S(t)$. Place all the elements previously computed on the graph.

We already did a big part of the graph, the one below has also the answers to the last question.

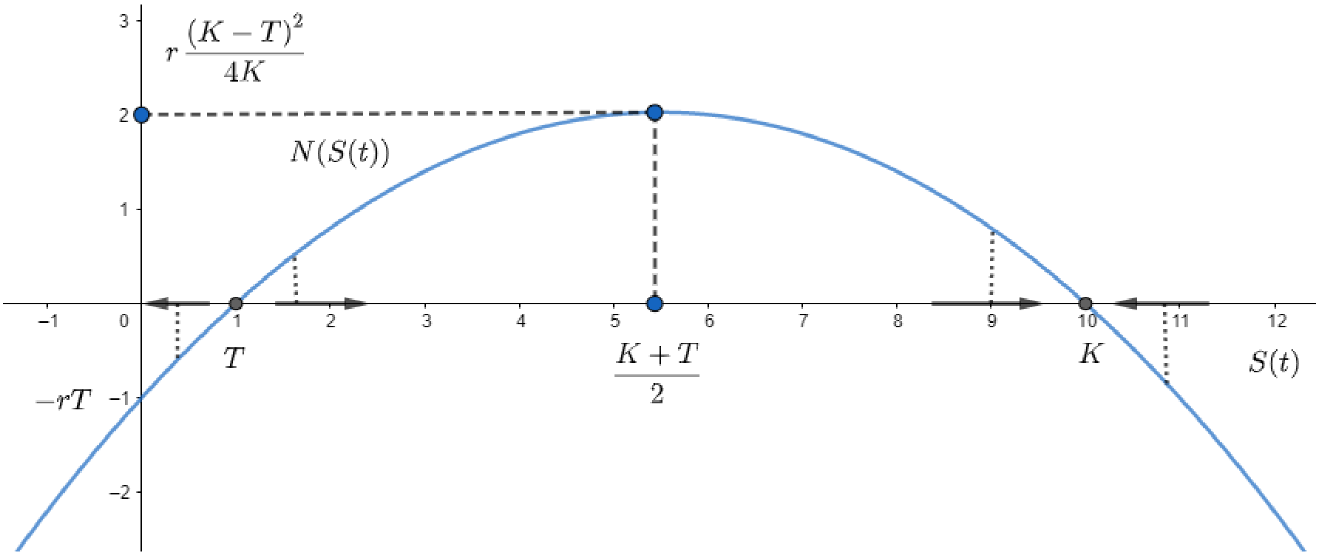

Figure 2: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$. Here $S^*(t) = \frac{10+1}{2} = 5.5$ and $N(S^*(t)) = \frac{9^2}{40} \approx 2$.

f. If there are B boats catching fishes, their total catches are $H(t) = \alpha BS(t)$. The net growth rate (the law of motion) of the stock is $\dot{S}(t) = N(t) - H(t)$. With B boats in the ocean, what is (are) the steady-state population(s) of fish?

First, notice that $\dot{S}_t$ is just notation for $\frac{\partial S(t)}{\partial t}$, the derivative of the stock of fish with respect to time. It is the equivalent of the law of motion of the Solow - Swan growth model, so you should treat it exactly as we did with that model. This observation helps us answering this question. In fact, the steady state population of fish is characterised by setting its growth rate equal to 0, which is the same as saying that $N(t) = H(t) - N(t) = 0 \implies H(t) = (t)$. We are looking for solutions of a quadratic equation, hence I rearrange terms to employ the classical formula.

$$\dot{S}_t = 0 \Leftrightarrow r(S(t) - T)\left(1 - \frac{S(t)}{K}\right) - \alpha B S(t) = 0$$

$$\Leftrightarrow rS(t) - rT - \frac{rS(t)^2}{K} - \frac{rTS(t)}{K} - \alpha BS(t) = 0$$

$$\Leftrightarrow -\frac{rS(t)^2}{K} + S(t)\left(r + \frac{rT}{K} - \alpha B\right) - rT = 0$$

$$\Leftrightarrow S(t)^2 \frac{r}{K} - S(t)\left(r + \frac{rT}{K} - \alpha B\right) + rT = 0$$

$$\Leftrightarrow S(t)^2 - S(t)\left(K + T - \frac{\alpha BK}{r}\right) + TK = 0 \text{ Multiply everything by } \frac{K}{r}$$

$$\Leftrightarrow S(t)^2 + S(t)\left(\frac{\alpha BK}{r} - K - T\right) + TK = 0$$

This is a second order degree equation of which we have to find the roots by the usual formula. There are two solutions that we label S_U and S_S (you will soon see why).

$$S_U = \frac{K + T - \frac{\alpha BK}{r} - \sqrt{\left(\frac{\alpha BK}{r} - K -\right)^2 - 4TK}}{2}$$

$$S_S = \frac{K + T - \frac{\alpha BK}{r} + \sqrt{\left(\frac{\alpha BK}{r} - K - T\right)^2 - 4TK}}{2}$$

These are the two steady state populations of fish.

g. Graph the dynamics of the stock with resource extraction and identify the equilibrium population(s) of fish. Show with arrows how population dynamics pushes S to increase or decrease.

Here the picture where I added the solutions computed in the previous point. Notice that here the growth rate is given by the difference $N(t) - H(t)$. Hence, to check the stability of steady states you have to see which one is above the other. Consider S_S in the picture, as an example. If you perturb it towards the right (i.e. $S_S + \varepsilon$), you see that $H(t)$ is greater than $N(t)$, hence $\dot{S}(t)$ is negative and we get back to S_S. By performing the same reasoning for any perturbation you can easily see that S_U is unstable while S_S is stable.

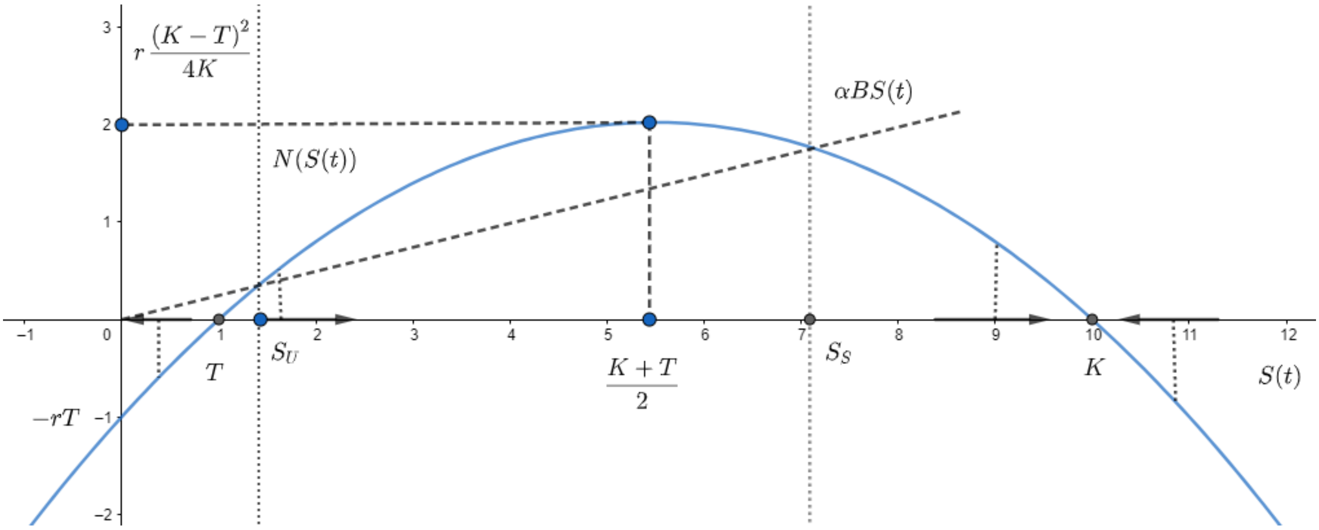

Figure 3: Same graph as before with S_U and S_S. Here I picked $\alpha = \frac{1}{8}$ and $B = 4$.

h. Is there an intensity of fishing (αB) so high that no sustainable fishing is possible? What is it?

To answer this question it is enough to notice that if you increase αB by a lot then the line $\alpha BS(t)$ will not cross $N(S(t))$ anymore, which means that no sustainable fishing is possible. This will happen when there is no solution to the previous second degree equation, that is when the quantity below the square root is negative (imaginary solution). Therefore we just have to check when this condition is satisfied. We search again for the solutions of a quadratic equation in αB.

$$\left(\frac{\alpha BK}{r} - K - T\right)^2 - 4TK < 0$$

$$\pm\left(\frac{\alpha BK}{r} - K - T\right) < \sqrt{4TK}$$

$$\frac{\alpha BK}{r} > T + K - 2\sqrt{KT} \quad \text{In the minus (-) case}$$

$$\alpha B > \frac{r}{K}\left(T + K - 2\sqrt{KT}\right)$$

$$\alpha B > \frac{r}{K}\left(\sqrt{T} - \sqrt{K}\right)^2$$

$$\alpha B > r\left(\frac{\sqrt{K}}{\sqrt{K}} - \frac{\sqrt{T}}{\sqrt{K}}\right)^2$$

$$\alpha B > r\left(1 - \sqrt{\frac{T}{K}}\right)^2$$

If you perform the same operation in the plus (+) case you will find that $\alpha B < r\left(1 + \sqrt{\frac{T}{K}}\right)^2$, hence we obtain that no sustainable fishing is possible when $r\left(1 - \sqrt{\frac{T}{K}}\right)^2 < \alpha B < r\left(1 + \sqrt{\frac{T}{K}}\right)^2$. Just as a remark, notice that in the numerical example from which I plotted the graph this condition is satisfied and therefore we have the two solutions.

Question: Can you guess what happens if $\alpha B > r\left(1+\sqrt{\frac{T}{K}}\right)^2$?

Question: Can you think about other methods to do this point?

i. The profit from a boat is $\pi(t) = p\alpha S(t) - c$. If there is free entry, fishing boats will enter as long as profits are positive. What is the free market equilibrium value of the stock S_F^* in the steady state.

If boats will continue to enter as long as profits are positive, then they will stop when profits are 0. Therefore, as in class, to find the free market equilibrium value of $S(t)$ we just need to check when this condition is satisfied.

$$\pi(t) = 0 \Leftrightarrow p\alpha S_F^* - c = 0 \Leftrightarrow S_F^* = \frac{c}{p\alpha}$$

However, notice that in class we had $T = 0$, and since $\frac{c}{p\alpha}$ is always weakly greater than 0 we never had any problem. In this case, if $\frac{c}{p\alpha} < T$ the growth is negative and the stock goes to 0.