TD6

4 Taxation to Obtain Optimum Resource Extraction

This exercise make you compute Pigouvian taxes. These kind of taxes are classical in the economics literature. Their aim is to correct for externalities that affect the market outcome without passing trough the channel of prices. In the fishing model an increase of boat affects the growth of natural resources in a way that is not transmitted to the market with the price p. Recall that the free market optimal number of boats is $B_F^* = \frac{r}{\alpha}\left(1 - \frac{c}{p\alpha K}\right)$, while from a social planner perspective we should have $B_O^* = \frac{B_F^*}{2}$.

a. Show that the optimal number of boats could be obtained by a tax per boat $t = \frac{p\alpha K - c}{2}$.

There are two ways to answer this question. The first way, which is the one you will have in the professor's solution, is to ask "which is the value of t such that if boats pay the new cost $c' = c + t$ then $B_O^* = B_{F'}^*$?". This means solving the following equation:

$$B_{F'}^* = \frac{r}{\alpha}\left(1 - \frac{c+t}{p\alpha K}\right) = \frac{r}{2\alpha}\left(1 - \frac{c}{p\alpha K}\right) = B_O^*$$

and realise that $t = \frac{p\alpha K - c}{2}$. Since you have this way already explained in your solution I will show you the second way. I think it is less smart but more algorithmic, in case you do not have the intuition to frame the problem in the way I just exposed.

The second way amounts to perform the same step you did in the class, but the profits are $\pi(t) = \alpha p S(t) - (c + t) = \alpha p S(t) - (c + \frac{p\alpha K - c}{2})$ and realise that the free market equilibrium boats are equal to the social optimum. In optimum we must always have that profits are equal to 0, therefore:

$$\alpha p S^*(B^*) - \left(c + \overbrace{\frac{p\alpha K - c}{2}}^{t} \right) = 0$$

$$\alpha p K \left(1 - \frac{\alpha B}{r}\right) - \left(c + \frac{p\alpha K - c}{2}\right) = 0$$

$$p\alpha K - \frac{p\alpha K \alpha B}{r} = c + \frac{p\alpha K - c}{2}$$

$$1 - \frac{c}{p\alpha K} - \frac{1}{2} + \frac{c}{2p\alpha K} = \frac{\alpha B}{r}$$

$$\frac{1}{2} - \frac{c}{2p\alpha K} = \frac{\alpha B}{r}$$

$$\frac{1}{2}\left(1 - \frac{c}{p\alpha K}\right)\frac{r}{\alpha} = B^*_{F'} = B^*_O$$

Which is the result we wanted.

b. Illustrate this tax on the graph of the revenue of the fishing industry.

The change in marginal revenues due to the introduction of the tax affects the point in which this line intersect the marginal cost c. You could also interpret it by saying that the new marginal cost is $c + t$ and the optimality condition requires the blue line to intersect with the new marginal cost.

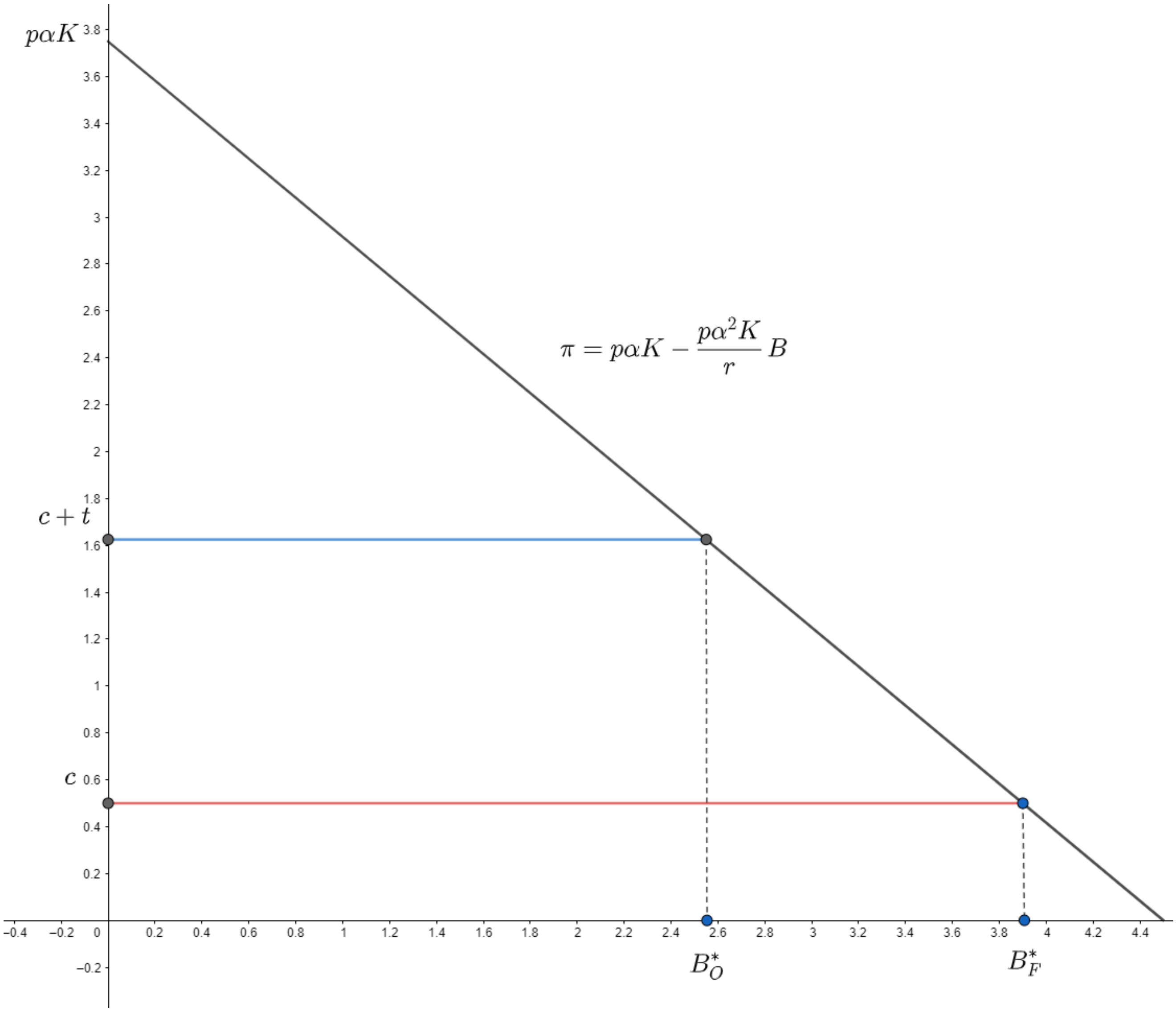

Figure 1: Profits and marginal costs for $p = 3$, $c = 0.5$ and, as before, $\alpha = \frac{1}{8}$, $K = 10$, $r = 1$.

c. Show that an ad valorem tax on fish sales of $\tau = \frac{p\alpha K - c}{p\alpha K + c}$ would achieve the optimum as well.

Exactly as before, we can solve this problem in two ways. The first one is to compute the τ proportional tax on π that would make $B_{F'}^* = B_0^*$. This is equivalent to solving the following equation:

$$B_{F'}^* = \frac{r}{\alpha}\left(1 - \frac{c}{p(1-\tau)\alpha K}\right) = \frac{r}{2\alpha}\left(1 - \frac{c}{p\alpha K}\right) = B_O^*$$

You have the detail of this method in the professor's solution.

The second method follows the same reasoning of the previous point. We just change the expression for profits and find the optimal number of boats in free markets $B_{F'}^*$. The new profits are $\pi(t) = p(1-\tau)\alpha S(t) - c = p\left(1 - \frac{p\alpha K - c}{p\alpha K + c}\right)\alpha S(t) - c$ and we must set them equal to 0.

$$p\left(1-\overbrace{\frac{p\alpha K-c}{p\alpha K+c}}^{\tau}\right)\alpha S(B^*)-c=0$$

$$p\left(1-\frac{p\alpha K-c}{p\alpha K+c}\right)\alpha K\left(1-\frac{\alpha B}{r}\right)-c=0$$

$$p\left(\frac{2c}{p\alpha K+c}\right)\alpha K\left(1-\frac{\alpha B}{r}\right)-c=0$$

$$\frac{p\alpha K2c}{p\alpha K+c}-\frac{p\alpha K2c\alpha B}{r(p\alpha K+c)}-c=0$$

$$\frac{p\alpha K2c}{p\alpha K+c}-c=\frac{p\alpha K2c\alpha B}{r(p\alpha K+c)}$$

$$1-\frac{\cancel{c}(p\alpha K+c)}{p\alpha K2\cancel{c}}=\frac{\alpha B}{r}$$

$$1-\frac{1}{2}-\frac{c}{p\alpha K2}=\frac{\alpha B}{r}$$

$$\frac{1}{2}\left(1-\frac{c}{p\alpha K}\right)\frac{r}{\alpha}=B^*_{F'}=B^*_O$$

Which is again the solution we wanted.

Question: How does the graph looks like here?

5 Equilibrium on Easter Island

In this exercise, we just have to reason with the phase diagram to get the answers. First, let's derive again the fundamental equations. Here we work in a system in which there is both growth of a natural resource, as in the fishing model, and population growth. As we just saw in the review question, the net growth of the natural resource is given by its natural growth rate minus the harvest (exactly as in the fishing model).

$$\begin{aligned}\dot{S}(t)&=\underbrace{rS(t)\left(1-\frac{S(t)}{K}\right)}_{\text{Natural growth}}-\underbrace{\alpha\beta S(t)L(t)}_{\text{Harvest}}\\&=\left(r\left(1-\frac{S(t)}{K}\right)-\alpha\beta L(t)\right)S(t)\end{aligned}$$

As for population growth, we can easily recover it by using the fundamental parameters of the model. We have that people die at a rate d and born at a rate b plus a percentage of per capita harvest $\phi\frac{H(t)}{L(t)}$. Of course, do not forget that the growth itself at time t depends on how many people $L(t)$ we have.

$$\begin{aligned}\dot{L}(t)&=\left(b-d+\phi\frac{H(t)}{L(t)}\right)L(t)\\&=\left(b-d+\phi\frac{\alpha\beta S(t)\cancel{L(t)}}{\cancel{L(t)}}\right)L(t)\\&=(b-d+\phi\alpha\beta S(t))L(t)\end{aligned}$$

These are the two **laws of motion** of the variables of interest in our model, $S(t)$ and $L(t)$. In the previous model with fish extraction we were used to graph the growth rate and see the points which constituted the stable states. Here the problem is a bit harder, as we have two variables with two laws of motion, not only one. Graphing the two growth rates separately will not help much in understanding the dynamics of the system. Hence, instead of employing a graph which on one axis has $S(t)$ and on the other one has its growth rate, we use a graph with the two variables of interest, both $S(t)$ and $L(t)$. But what do we represent?

We have a system with two variables and two growth rates, which means that each law of motion will have more than one couple of $S(t)$ and $L(t)$ in which it is zero. Take $\dot{S}(t)$, as an example. In the previous exercise we found the value of $S(t)$ (the fish stock) for which its growth was equal to 0. We do the same here, but we will not find a value of $S(t)$, rather a relation between $S(t)$ and $L(t)$! Let's do it, when is $\dot{S}(t)$ equal to 0?

$$\dot{S}(t) = 0 \Leftrightarrow \left(r\left(1 - \frac{S(t)}{K}\right) - \alpha\beta L(t)\right)S(t) = 0$$

This equation has two solutions: the first is $S(t) = 0$, for any value of $L(t)$, while the second is $r\left(1 - \frac{S(t)}{K}\right) - \alpha\beta L(t) = 0$, which gives us $S(t) = K - \frac{\alpha\beta K}{r}L(t)$. You see now that $\dot{S}(t)$ is not zero only for some values of $S(t)$, but for couples of values of $S(t)$ and $L(t)$. In all the points of the graph with axis $(L(t), S(t))$ in which $S(t) = K - \frac{\alpha\beta K}{r}L(t)$, the law of motion of S is equal to 0.

We must now repeat the same exercise with $\dot{L}(t)$.

$$\dot{L}(t) = 0 \Leftrightarrow (b - d + \phi\alpha\beta S(t))L(t)$$

We have two solutions here too: the law of motion is null when $L(t) = 0$, for any value of $S(t)$, but also when $b - d + \phi\alpha\beta S(t) = 0$ which gives $S(t) = \frac{d-b}{\phi\alpha\beta}$.

Hence, the equations that give us the set of points $S(t)$ and $L(t)$ in which the two law of motions are zero are given by:

$$S^* = K - \frac{K\alpha\beta}{r}L^* \qquad and \qquad S^* = 0$$

$$S^* = \frac{d-b}{\phi\alpha\beta} \qquad and \qquad L^* = 0$$

When our variables respect these conditions there is no movement of S (first equation) or L (second equation). The system completely stops when we are in the steady state, which in this case is when both these equations are satisfied, i.e. neither S nor L are moving. Every time we are not in a point in which these two lines intersect we have movement in the system. If we forget about the 0 solutions we can graph the two lines with the dynamics when we are out of the steady state here.

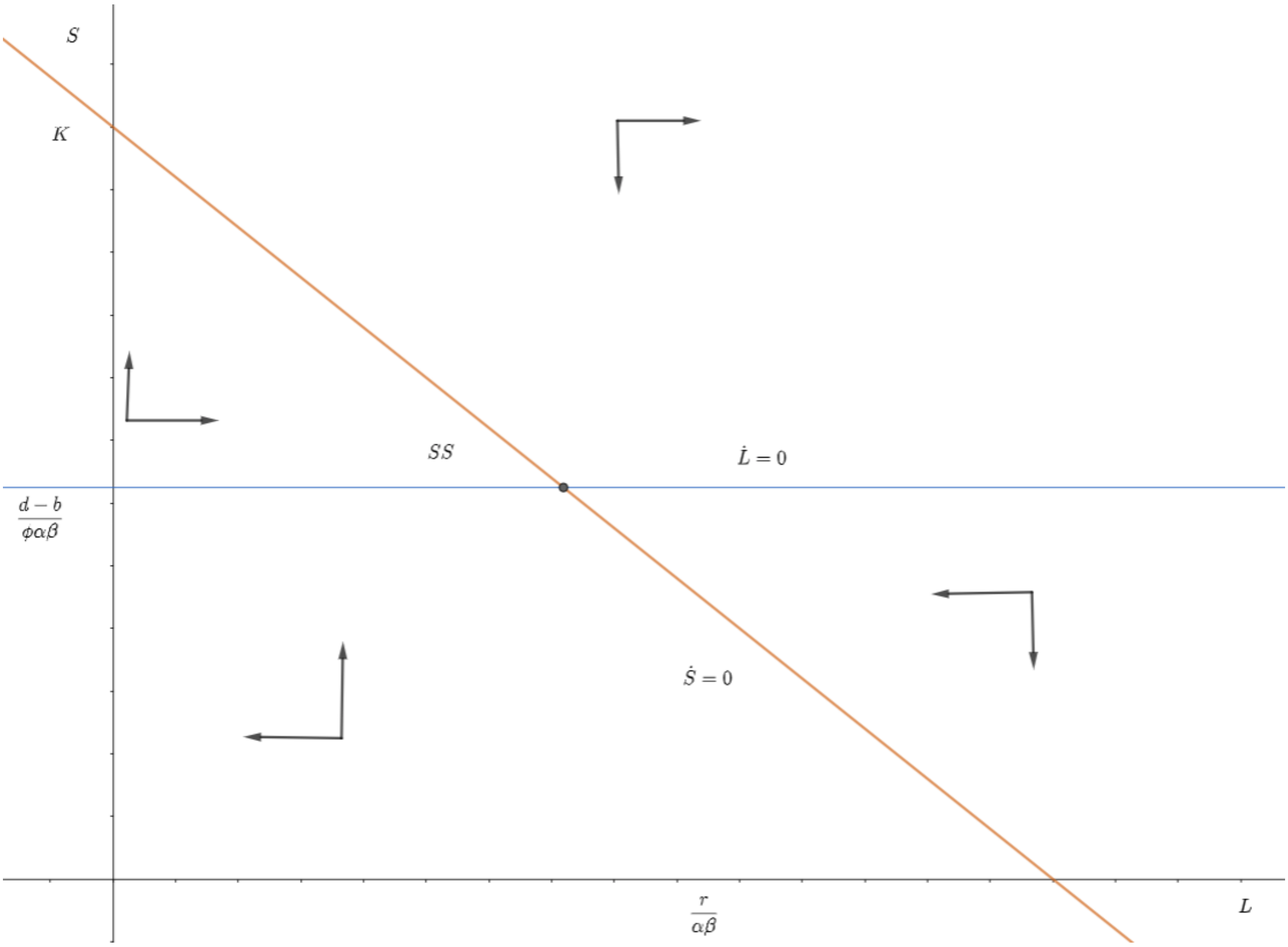

Figure 2: Phase diagram with $r = 0.04$, $\alpha = 10^{-6}$, $b - d = 0.1$, $\phi = 4$, $\beta = 0.4$, $K = 12000$.