TD8

1 Review

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a. Before the demographic transition, increases in income per capita always caused an increase in the growth rate of population.

- *Answer*

  True: As you found in your lecture, in the pre industrialised world technological progress and land expansion caused temporary increase in income per capita which in turn increased the growth rate of population. As an example see the graph at page 3.

b. In the contemporary world, an increase in income per capita is associated to a decrease in the growth rate of population.

- *Answer*

  True: The graph at page 12 of your lecture notes is self explanatory. Higher average income correlates with lower births per woman. An explanation taken from your lecture notes could be the effects of urbanization.

c. Decreases in the various measures of fertility came after decreases in mortality.

- *Answer*

  True: You can see from the graph at page 11 of your lecture notes that the fertility rate has decreased from 1950. For sure the mortality rate had significantly declined when compared to pre 1900 times. This statement is also consistent with the fact that fertility rates are lower in developed countries where the mortality rate is lower.

d. The demographic transition is now over for most of the world population.

- *Answer*

  True: Quoting from page 11 of your notes "*as of 2017, more than 80% of the world fertility was already at or below replacement rate*".

e. In the model of the Malthusian regime seen in class, an exogenous increase in the birth rate translates into a lower level of steady-state income per capita.

- *Answer*

  Maybe: If this shock implies that $b(y_t)$ is higher, keeping other things fixed, then the statement is true, you can see it from the graph presented here. Clearly, if the shock is such that $b\left(y_t\right)$ is lower, then income per capita will be higher.

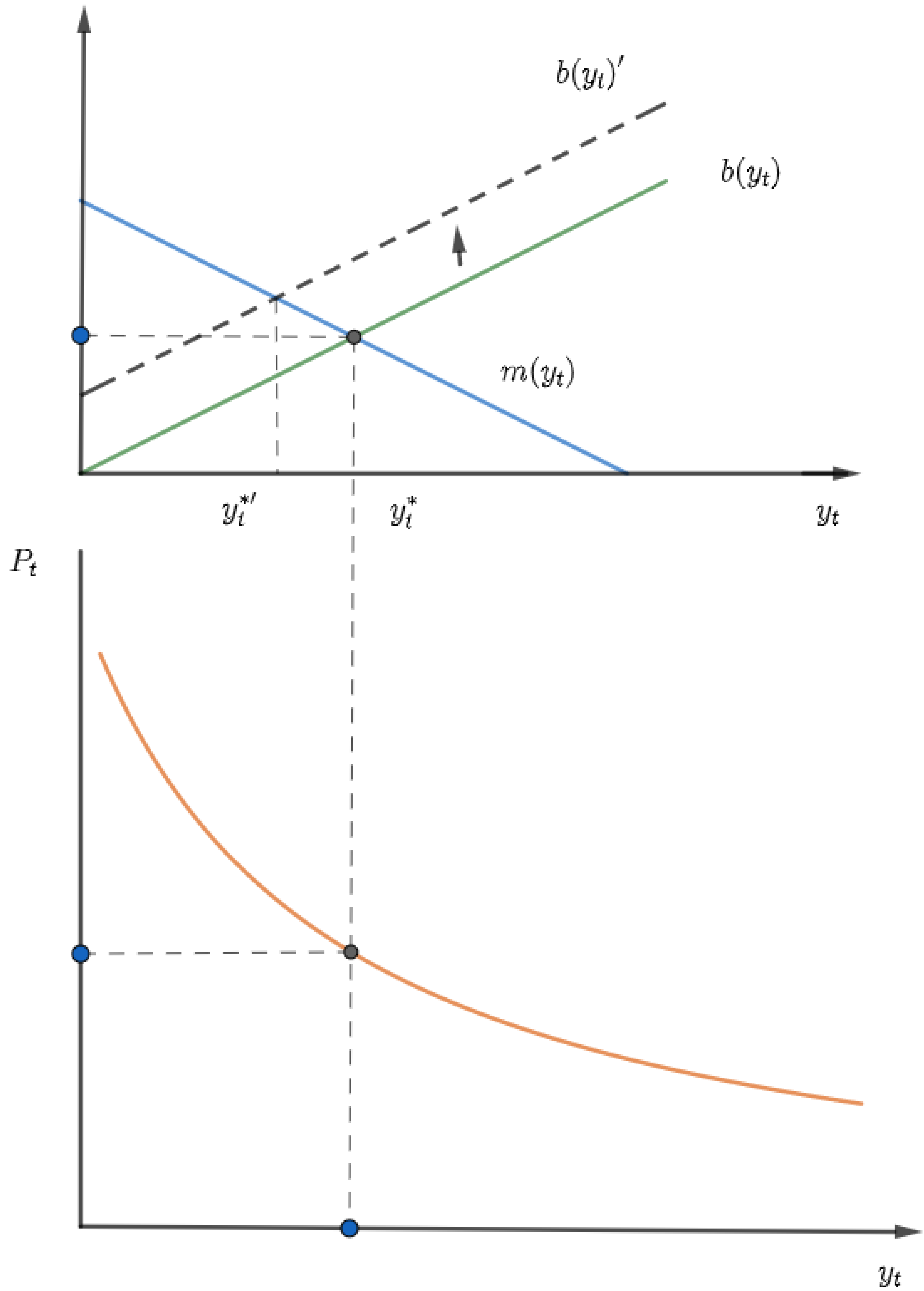

Figure 1: Shock to $b\,(y_t)$.

2 The Malthusian Regime

This exercise studies a particular example of the model seen in class, with specifications for the primitives of the model. I report them here.

The birth rate is given by:

$$b(y_t) = \alpha_b + \beta_b y_t \tag{1}$$

The mortality rate is:

$$m(y_t) = \alpha_m - \beta_m y_t \tag{2}$$

The production function is:

$$Y(P_t) = \alpha_y + \beta_y P_t \tag{3}$$

Y is the total production (or income), P is the population, y is income per capita. It is assumed that all the population works and that there is no immigration nor emigration. All coefficients of the model are positive. It is further assumed that $\alpha_m > \alpha_b$ and $\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m) > 0$, you will soon understand why.

a. Discuss equations (1) and (3): how do they relate to the model seen in the lecture?

In your lectures you saw that the birth rate depends positively on income per capita, which translates into $b(y_t)$ with $\frac{\partial b(y_t)}{\partial y_t} > 0$. This condition is indeed respected in the specific function we have in this exercise, as $\frac{\partial b(y_t)}{\partial y_t} = \beta_b > 0$.

As for the technology, the condition you had in the lecture were positive marginal product $\frac{\partial Y(P_t)}{\partial P_t} > 0$ and decreasing returns $\frac{\partial^2 Y(P_t)}{\partial P_t^2} < 0$. It is easy to check that the first holds while the second does not. In fact, $\frac{\partial Y(P_t)}{\partial P_t} = \beta_y > 0$ and $\frac{\partial^2 Y(P_t)}{\partial P_t^2} = 0 \nless 0$. This production function is similar to what you saw in TD2 in the Solow Model, it is affine! Since it is a line it does not have decreasing returns. Will this be a problem for the existence of a steady state? Let's see...

b. Compute the marginal and average productivity of labor. Comment.

To compute average and marginal productivity, we first need productivity. As usual we divide by the population P_t.

$$\begin{aligned} y(P_t) &= \frac{Y(P_t)}{P_t} \\ &= \frac{\alpha_y + \beta_y P_t}{P_t} \\ &= \frac{\alpha_y}{P_t} + \beta_y \end{aligned}$$

Marginal productivity indicates how much more productive we are by increasing P_t by an infinitesimal amount. By taking the derivative we get:

$$\frac{\partial y(P_t)}{\partial P_t} = -\frac{\alpha_y}{P_t^2} < 0$$

Hence, by adding labour we become less productive. As for average productivity we just have to divide by the number of workers

$$\frac{y(P_t)}{P_t} = \frac{\alpha_y}{P_t^2} + \frac{\beta_y}{P_t}$$

Nothing special here, as we increase the number of workers the productivity per worker decreases.

c. Compute the steady-state level of total income, per capita income and population. Show graphically how those steady-state values are determined.

The law of motion of population in this model is $\dot{P} = [(b(y_t) - m(y_t))]P$. As usual, we are in steady state when the law of motion is equal to 0, i.e. population is not growing $\dot{P} = 0$. This when the mortality rate is equal to the birth rate (actually, also when $P = 0$, but we are not interested in this case). We have to set $b(y_t) = m(y_t)$ to get:

$$\begin{aligned} \alpha_b + \beta_b y^* &= \alpha_m - \beta_m y^* \\ y^*(\beta_b + \beta_m) &= \alpha_m - \alpha_b \\ y^* = \frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} \end{aligned}$$

which is the steady state level of per capita income. You should see now why we assumed $\alpha_m > \alpha_b$. We still need to compute the steady-state level of total income and population. Total income is itself a function of the population, so we have to find the steady state of this one first. We can rely on y^* which we just found.

$$\begin{aligned} y^* &= \frac{\alpha_y}{P^*} + \beta_y \\ \frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} &= \frac{\alpha_y}{P^*} + \beta_y \\ \frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} - \beta_y &= \frac{\alpha_y}{P^*} \\ P^*\left(\frac{\alpha_m - \alpha_b}{\beta_m + \beta_b} - \beta_y\right) &= \alpha_y \\ P^*\left(\frac{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}{\beta_m + \beta_b}\right) &= \alpha_y \\ P^* &= \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} \end{aligned}$$

We obtain a positive P^* since we assumed $\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m) > 0$. Now that we have y^* and P^* we are ready to compute the steady state level of total income Y^*. Its expression is given in the text:

$$
\begin{aligned}
Y^* = Y(P^*) &= \alpha_y + \beta_y P^* \\
&= \alpha_y + \beta_y \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} \\
&= \frac{\alpha_y\alpha_m - \alpha_y\alpha_b - \cancel{\alpha_y\beta_y\beta_b} - \cancel{\alpha_y\beta_y\beta_m} + \cancel{\alpha_y\beta_y\beta_b} + \cancel{\alpha_y\beta_y\beta_m}}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} \\
Y^* &= \frac{\alpha_y(\alpha_m - \alpha_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}
\end{aligned}
$$

In the following figure I represented the equilibrium for specific values of the parameters.

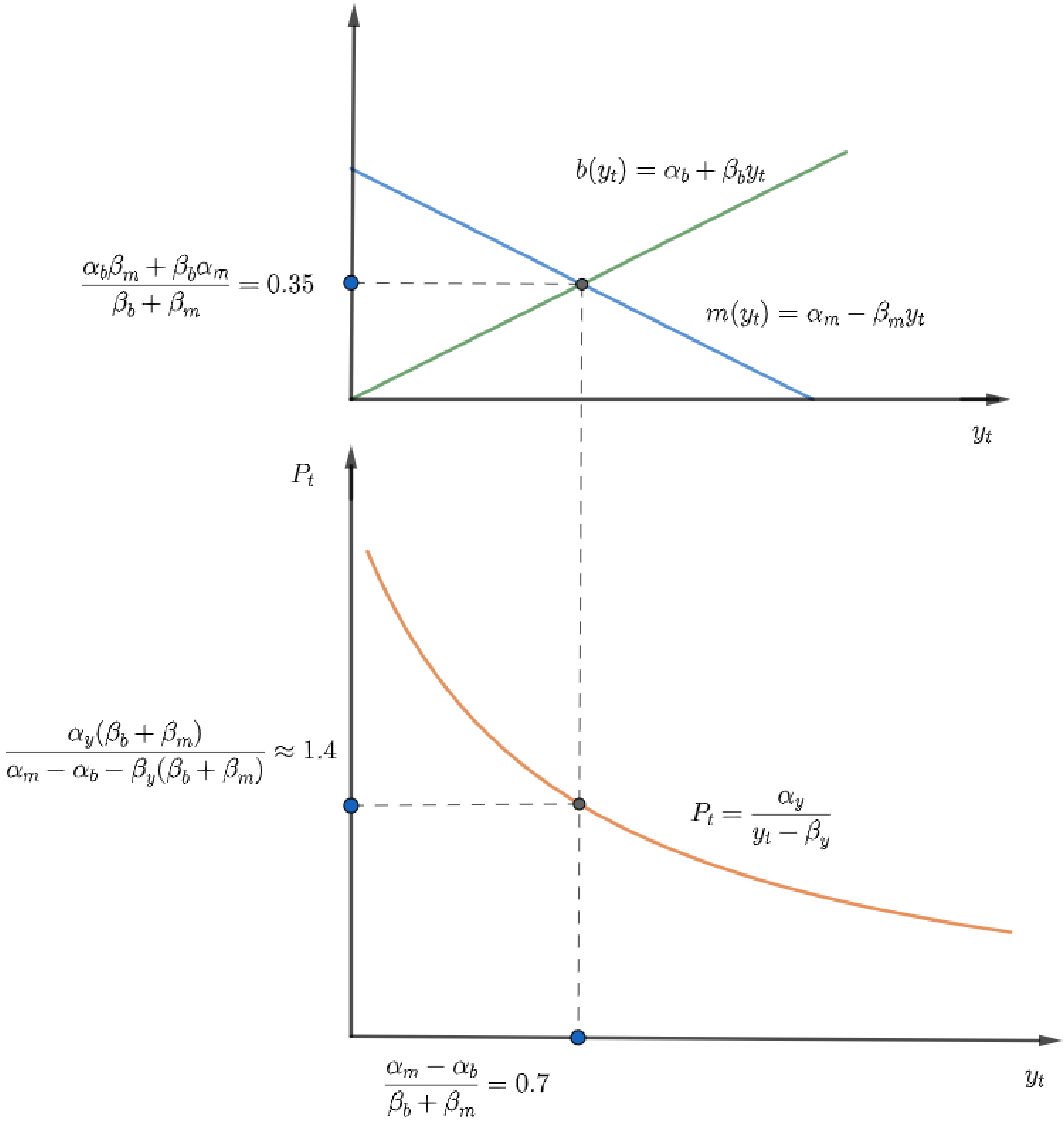

Figure 2: Steady state for $\alpha_b = \beta_y = 0$, $\beta_b = \beta_m = 0.5$, $\alpha_y = 1$ and $\alpha_m = 0.7$. Here $Y^* = \alpha_y + \beta_y P^* = 1$.

Question: Can you guess what happens if $\alpha_m - \alpha_b < 0$?