TD9

2 The Malthusian Regime

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d. For the rest of the exercise, we assume $\alpha_b = \beta_y = 0$, $\beta_b = \beta_m = 0.5$, $\alpha_y = 1$ and $\alpha_m = \alpha > 0$. What are the steady-state levels for this configuration of parameters?

I more or less computed them in the graph, but I assumed a specific value for α, let's do it again. We have $Y^* = \alpha_y + \beta_y P^* = 1 + 0P^* = 1$. As for y^* and P^*:

$$y^* = \frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} = \frac{\alpha}{1} = \alpha$$

$$P^* = \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} = \frac{1}{\alpha}$$

So $P^* = \frac{1}{\alpha}$, $y^* = \alpha$ and $Y^* = 1$.

e. Show that the model dynamics can be summarized by a first-order difference equation in P_t (of the type $P_{t+1} = f(P_t)$, with f some function that you need to find; you can also look for an equation of the type $\Delta P_t = g(P_t)$ with g some function to find, if it is easier for you to do so).

This question is a very involved way of asking: what are the time dynamics of P_t? You know from your lecture notes that $\dot{P} = [b(y_t) - m(y_t)]P_t$. However, we are in discrete time here, as the question asks for a difference equation (not differential), therefore in this case $\dot{P}$ is substituted by $P_{t+1} - P_t$. We just have to work out the expression above and plug values for the parameters.

$$\begin{aligned}
P_{t+1} - P_t &= [b(y_t) - m(y_t)]P_t \\
&= \left[\cancel{\alpha_b} + \beta_b y_t - \alpha_m + \beta_m y_t\right]P_t && \text{since } \alpha_b = 0 \\
&= [(\beta_b + \beta_m)y_t - \alpha]P_t && \text{since } \alpha_m = \alpha \\
&= [y_t - \alpha]P_t && \text{since } \beta_b + \beta_m = 1 \\
&= \left[\frac{\alpha_y}{P_t} + b_y - \alpha\right]P_t && \text{substituting } y_t(P_t) \\
&= \left[\frac{1}{P_t} - \alpha\right]P_t && \text{since } \alpha_y = 1 \text{ and } \beta_y = 0 \\
P_{t+1} - P_t &= 1 - \alpha P_t \\
P_{t+1} &= P_t(1 - \alpha) + 1
\end{aligned}$$

f. Study the convergence of population to its steady state starting from an initial value of population P_0 close to 0 for the following values of α: (i) $0 < \alpha < 1$, (ii) $\alpha = 1$, (iii) $1 < \alpha < 2$.

This question basically asks you to study the dynamics of population for different values of α. It is more or less about plugging numbers. Let's start from $t = 1$ and see what the dynamics look like. Since $1 - \alpha$ is a bit uncomfortable I substitute it with $\gamma = 1 - \alpha$. Let's start easy and substitute numbers time by time.

$$\begin{aligned} P_1 &= \gamma P_0 + 1 \\ P_2 &= \gamma P_1 + 1 \\ &= (\gamma P_0 + 1)\gamma + 1 \\ &= \gamma^2 P_0 + \gamma + 1 \\ P_3 &= \gamma P_2 + 1 \\ &= (\gamma^2 P_0 + \gamma + 1)\gamma + 1 \\ &= \gamma^3 P_0 + \gamma^2 + \gamma + 1 \end{aligned}$$

You see the pattern. By thinking a little bit you should realise that we can express P_t in the following way:

$$P_t(\gamma) = \gamma^t P_0 + \sum_{s=0}^{t-1} \gamma^s$$

For $t \to \infty$, by the rules of power series, we have:

$$\begin{aligned} P_\infty(\gamma) &= \gamma^\infty P_0 + \sum_{s=0}^{\infty} \gamma^s \\ &= \gamma^\infty P_0 + \frac{1}{1-\gamma} \end{aligned}$$

We are ready to evaluate the convergence. The following table gives a relationship between $1 - \alpha$ and γ .

	α	γ
(i)	$0 < \alpha < 1$	$0 < \gamma < 1$
(ii)	1	0
(iii)	$1 < \alpha < 2$	$-1 < \gamma < 0$

Case (ii) is the easiest. If $\gamma = 0$ then $P_t = 1$ for any t. Population is fixed since the beginning, so in some sense we already converged from the start to 1.

In case (i) we have $0 < \gamma < 1$. If we have no clue we can take one number and see what happens. Let's try $\gamma = 0.5$. We have the following series (assuming P_0 is close to 0):

$$
\begin{aligned}
P_1 &= 0.5P_0 + 1 \approx 1 \\
P_2 &= 0.5(1) + 1 = 1.5 \\
P_3 &= 0.5(1.5) + 1 = 1.75 \\
P_4 &= 0.5(1.75) + 1 = 1.875 \\
P_5 &= 1.9375 \\
P_6 &= 1.96875 \\
&\vdots
\end{aligned}
$$

In the following picture you can see the series graphically:

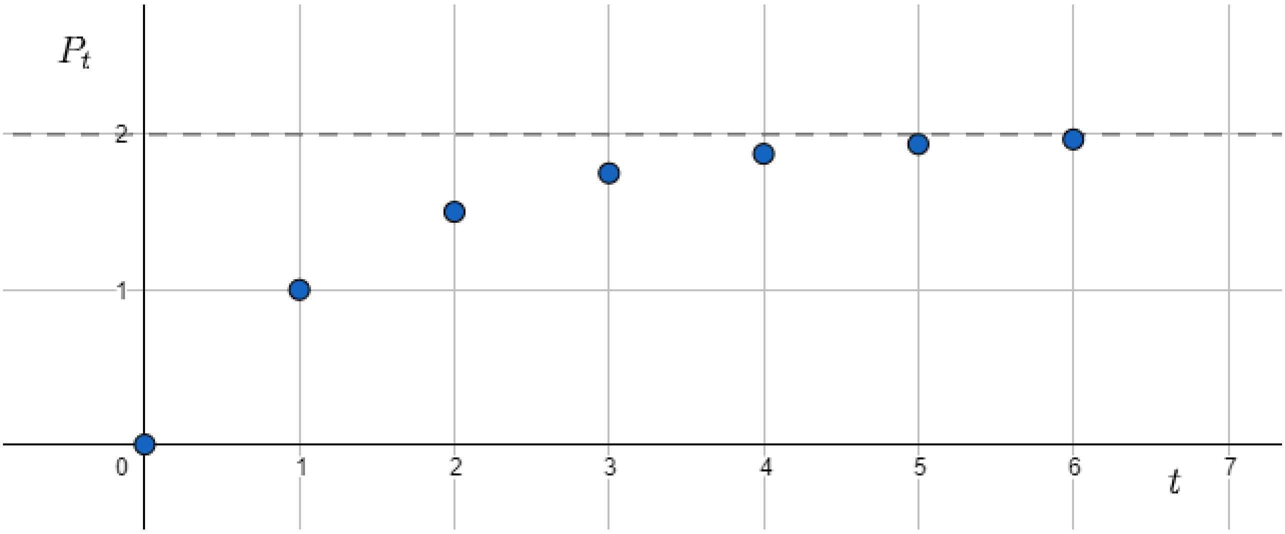

Figure 1: Series with $\gamma = 0.5$.

You can see where we are going. We can immediately compute P_∞ from the expression above (remember that for any $-1 < \gamma < 1$ we have that $\gamma^\infty = 0$):

$$P_\infty(0.5) = 0P_0 + \frac{1}{1-0.5} = \frac{1}{0.5} = 2$$

What we conclude is that for a value of α between 0 and 1 population grows, slower at each step, and eventually reaches a steady state level (different for different values of γ).

As for case (iii) , with $-1 < \gamma < 0$, we use the same strategy, namely plugging numbers for the specific value $\gamma = -0.5$. The series looks like this:

$$
\begin{aligned}
P_1 &= -0.5P_0 + 1 \approx 1 \\
P_2 &= -0.5(1) + 1 = 0.5 \\
P_3 &= -0.5(0.5) + 1 = 0.75 \\
P_4 &= -0.5(0.75) + 1 = 0.625 \\
P_5 &= 0.6875 \\
P_6 &= 0.65625 \\
P_7 &= 0.671875 \\
&\vdots
\end{aligned}
$$

As before, I plotted the series:

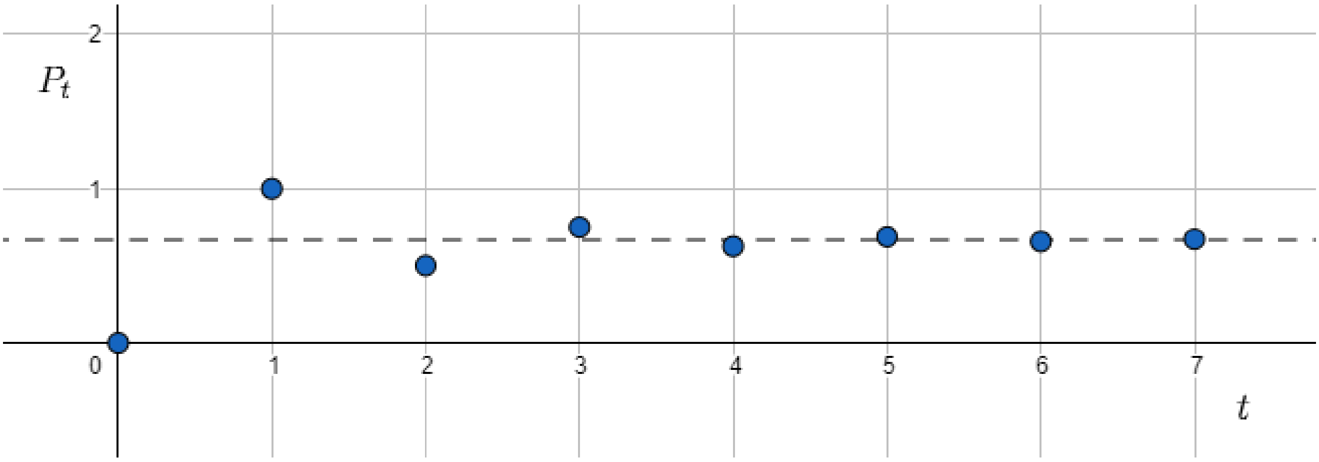

Figure 2: Series with $\gamma = -0.5$.

As you can see the series here goes up and down, it is not monotonic in its growth, contrary to the previous case. However, we can see where it converges to:

$$P_\infty(-0.5) = 0P_0 + \frac{1}{1-(-0.5)} = \frac{1}{1.5} = 0.\bar{6}$$

Interestingly, notice that the term P_0 has in both case no role in determining the convergence, only shaped by γ.